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Aumann-Shapley Random Order Values of Non-Atomic Games

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Abstract. In this paper we report results for a reformulated random order approach to values of non-atomic games. The reformulation is achieved by generating random orders from a fixed subgroup of automorphisms Θ that admits an invariant probability measurable group structure. The resulting Θ-symmetric random order value operator is unique and satisfies all the axioms of a Θ-symmetric axiomatic value operator. For the uncountably large invariant probability measurable group $\left(\breve{\Theta}, \breve{\mathcal{B}}, \breve{\Gamma}\right)$ of Lebesgue measure preserving automorphisms constructed in Raut [1997], $\breve{\Theta}$-symmetric random order value exists for most games in BV and it coincides with the fully symmetric Aumann-Shapley axiomatic value on large subspaces of games in pNA. Thus by restricting the set of admissible orders and the space of games suitably, it is possible to circumvent the Aumann-Shapley Impossibility Principle for the random order approach to values of non-atomic games.

1 Introduction

Given a fixed set of players, called the grand coalition, and an algebra of its subsets as the possible coalitions, a cooperative game is a set function that assigns a real number to a coalition, called the worth of the coalition, to each coalition. The worth of a coalition is the payoff which the players can earn cooperatively. A basic problem in cooperative game theory is to find rules, called solution concepts, for dividing the worth of the grand coalition among the players so that certain desiderata are satisfied. Mathematically, the problem is to find a mapping or an operator satisfying pre-specified conditions from the space of all set functions to the space of additive set functions. The Shapley value is one such solution concept. The axiomatic characterization provided by Shapley is typically viewed as expressing fairness properties; these may account for its wide commendation.

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Using the linear vector space structure of the space of games, Shapley [1953] proved the existence and uniqueness of the operator satisfying his axioms and provided a formula to compute the value of individual games. The solution thus obtained is known as *axiomatic value.* Shapley also postulated an alternative set of fairness properties which come to be known as the *random order value.* In this approach, a player is given his expected marginal contribution in a random ordering of players, each ordering being equally likely among all possible orderings of the players. Shapley showed that the formulas for value from both approaches coincide.

Aumann and Shapley [1974] extended the concept of axiomatic value to games with a continuum of players and proved the existence and uniqueness of an axiomatic value operator on the spaces of games such as pNA and bv′ NA (definitions of unknown terms in the introduction can be found in subsequent sections). They also provided a "diagonal formula" for games in pNA. In an attempt to extend the random order value to continuum case, Aumann and Shapley considered the set Ω of all orderings of players satisfying some measurability condition and derived an impossibility principle: *There does not exist a measure structure on Ω with respect to which a random order value could be assigned to games in pNA.*

There have been several developments in the axiomatic value over the past several years. One direction of research has been aimed at finding larger spaces of games on which an axiomatic value, possibly a unique one, exists. The space pNA is economically the most important one since it contains smooth market games and fair cost allocation schemes. However, non-smooth games that arise from markets with strong complementarity do not belong even to the space ASYMP.[1] Mertens [1988] introduced a very large space of games, known as *Mertens space.* This space includes these non-smooth games and also all the above spaces. Using his extension of the diagonal formula Mertens proved the existence of an axiomatic value on Mertens space.

The most important fairness properties of the Shapley value are derived from the symmetry axiom. This axiom was originally specified with respect to the whole group $\mathcal{G}$ of automorphisms of the players set. More generally, we view each axiomatic value with symmetry restricted to a subgroup $\Theta \subset \mathcal{G}$ as representing a particular fairness principle. We shall refer to a value with symmetry restricted to a subgroup Θ as *Θ-symmetric axiomatic value* and the mapping which assigns to each game its Θ-symmetric axiomatic value as a *Θ-symmetric axiomatic value operator.* Some economic applications may dictate the use of particular subgroups Θ. For a given non-atomic measure μ on the unit interval, let us denote by $\mathcal{G}(\mu)$ the group of μ-measure preserving automorphisms. Monderer [1986, 1989] showed that the non-atomic games that arise from smooth market economies have certain characteristics in which symmetry group could be restricted to the group $\mathcal{G}(\mu)$ and the non-atomic measure μ is determined by the data of the economy. For such applications the domain of the value operator can be restricted to the space pNA(μ), the linear subspace of pNA generated by the polynomials in non-atomic probability measures which are absolutely continuous with respect to the above measure μ.

[1]The space ASYMP contains the space pNA and it is the largest space on which value was shown to exist in the Aumann and Shapley [1974] monograph. The technique used to prove this result (see Kannai [1966] and also Aumann and Shapley [1974] for details) was to approximate a continuum game with a sequence of finite games and then to use a limiting argument on the sequence of Shapley values of these finite games to arrive at the Shapley value of the continuum game. The space ASYMP consists of games for which this limiting argument was valid.

Monderer showed that any $\mathcal{G}(\mu)$-symmetric axiomatic value operator on pNA(μ) is also symmetric with respect to the full group of automorphisms $\mathcal{G}$.

Aumann and Shapley [1974] proved that there does not exist $\mathcal{G}$-symmetric axiomatic value operator on all of BV. Thus to have a value on all of BV, the symmetry axiom must be restricted to a proper subgroup. Ruckle [1982] has shown that when the symmetry is restricted to any "locally finite" subgroup of automorphisms, there exists a value operator on all of BV. This result is further refined by Monderer and Ruckle [1990].

It is important to note that the main fairness property of the random order value arises from the fact that each player has an equal chance of forming a coalition with a set of players of any size and any names. The random order value assigns to each player the average of his marginal contributions over all coalitions which he may join. In the finite player case, the group of automorphisms of the players set and the set of orderings of players generated by them are identical, and the unweighted averaging of the marginal contributions of a player over all orderings generated by a given group of automorphisms basically symmetrizes the random order value for the group of automorphisms. Raut [1993] shows that for the average marginal contribution to be symmetric for all games and all players it is necessary and sufficient that the randomness of the set of orders that are induced by the symmetry group assigns equal likelihood to each order, i.e., a random order has uniform distribution. I used these insights from finite games to reformulate a Θ-symmetric random order value for continuum case (Raut [1995, 1997]), which can be briefly described as follows.

We begin with a fixed group of automorphisms $\Theta \subset \mathcal{G}$ with an invariant probability measurable group structure in it. We take the set of orders Ω as those which are determined by Θ and induce a uniform distribution on Ω from the action of the invariant probability measurable group Θ. We first show that the expected value of the marginal contribution function on Ω is unique (i.e., independent of any particular invariant measurable group structure on Θ) and it satisfies all the axioms of Θ-symmetric value; we name such a mathematical expectation operator as Θ-*symmetric random order value operator* on any subspace of games on which it is well defined. In particular then the $\mathcal{G}$-symmetric random order value operator is also a $\mathcal{G}$-symmetric axiomatic value operator. The set of orders induced by $\mathcal{G}$ is, however, a proper subset of the set of Aumann-Shapley measurable orders (see Raut [1995, Proposition 1] for details). Could we not then hope to have a possibility result for the Aumann-Shapley Impossibility Principle with respect to the $\mathcal{G}$-symmetric random order value operator? We do not have an answer to this question yet. However, with the help of even more restricted set of orders induced from a suitably constructed uncountably large invariant probability measurable group $\breve{\Theta}$ of Lebesgue measure preserving (l.m.p.) automorphisms, we have proved a partial possibility theorem to the Aumann-Shapley Impossibility Principle. More specifically, we have shown that there exists a unique $\breve{\Theta}$-symmetric random order value operator on a large space of games (we named it as NBV) which contains most of the games in BV and some of the non-smooth games that did not belong even to ASYMP. For any non-atomic probability measure μ, let us define the space lpNA(μ) as the linear space of games generated by the polynomials in μ. Most economic applications consider games in pNA which also belong to lpNA(μ) for some non-atomic probability measure μ. We have also derived a diagonal formula for the $\breve{\Theta}$-symmetric random order value which coincides with the diagonal value formula for the $\mathcal{G}$-symmetric Aumann and Shapley axiomatic value for games in the space lpNA(μ). Thus, when the set of admissible orders Ω are suitably restricted we have a measure structure on Ω with respect to which the random order value exists for large subspaces of games in pNA.

In Section 2 we lay out the basic framework for our random order approach, discuss the relevant issues and state the main results. In Section 3, we discuss the issues concerning choice of a suitable symmetry group, and sketch the construction of invariant probability measurable group $\breve{\Theta}$ which was introduced in Raut [1997]. Finally in Section 4 we put together further remarks concerning the random order approach to values for future research.

2 The basic framework

To denote a Borel σ-algebra of a topological space X (i.e., the σ-algebra generated by the class of open sets of X), we use the subscripted notation $\mathcal{B}_X$ and to denote any general σ-algebra, we do not use a subscript. Let $I = [0,1] \subset \Re$ be the set of players. Let $\mathcal{B}_I$ be the Borel σ-algebra of I. The elements of $\mathcal{B}_I$ are the set of admissible coalitions. A *game* is a set function $V : \mathcal{B}_I \longrightarrow \Re$ such that $V(\emptyset) = 0$. Let G_I be the set of all games. Let FA be the set of finitely additive set functions on $(I, \mathcal{B}_I)$. A *measure* is a countably additive set function. One can check easily that G_I and FA are linear vector spaces. A game V is *monotonic* if $V(S) \leq V(T)$ for any $S, T \in \mathcal{B}_I$, $S \subset T$. A *Borel automorphism* is a measurable map $\theta : (I, \mathcal{B}_I) \longrightarrow (I, \mathcal{B}_I)$ such that it is one-one, onto and θ^{-1} is also measurable. Let $\mathcal{G}$ be the set of all Borel automorphisms on $(I, \mathcal{B}_I)$. One can check that $\mathcal{G}$ is a non-commutative (also known as non-abelian) group with composition of functions as group multiplication operation and identity function as the group identity.

For each $\theta \in \Theta$, define the linear operator $\theta^* : G_I \longrightarrow G_I$ by

$$(\theta^* V)(S) = V(\theta^{-1}(S)),\ \forall S \in \mathcal{B}_I.$$

Given a subgroup of automorphisms, $\Theta \subset \mathcal{G}$, a linear subspace $Q \subset G_I$ is said to be Θ-*symmetric* if $\theta^* Q \subset Q$ for all $\theta \in \Theta$.

Let Q be a linear subspace of G_I. An operator $\Phi : Q \longrightarrow FA$ is said to be *linear* if $\Phi(\alpha V_1 + V_2) = \alpha\Phi(V_1) + \Phi(V_2)\ \forall\ V_1, V_2 \in Q, \alpha \in \Re$. The operator Φ is said to be *positive* if (ΦV) is monotonic for any monotonic V in the domain of Φ, and *efficient* if $\Phi V(I) = V(I)\ \forall\ V \in Q$. For a Θ-symmetric space Q, the operator $\Phi : Q \longrightarrow FA$ is said to be Θ-*symmetric* if $\Phi\theta^* V = \theta^*\Phi V,\ \forall\ \theta \in \Theta, V \in Q$.

A Θ-*symmetric axiomatic value operator* on a Θ-symmetric linear space of games Q is a positive, linear, efficient, and Θ-symmetric operator $\Phi : Q \longrightarrow FA$. When Θ is the full group $\mathcal{G}$, this operator is the *Aumann-Shapley axiomatic value operator*. Aumann and Shapley [1974] proved the existence and uniqueness of this operator axiomatically.

Although in our analysis of random order value we do not use any topological structure on the space of games, to relate our results to the literature, we adopt the following topological concepts from Aumann and Shapley [1974]. A game V is *of bounded variation* if there exist monotonic games U and W such that $V = U - W$. Denote by BV the set of all games of bounded variation. It is known that BV is a linear vector space over $\Re$. Define a map $\| \,.\, \|_{BV}$: BV $\longrightarrow \Re$ by

$$\| V \|_{BV} = \inf\ \{U(I) + W(I) \mid V = U - W,\ U \text{ and } W \text{ are monotonic games}\}$$

for each $V \in$ BV. It can be shown that $\| \,.\, \|_{BV}$ is a well defined norm, on BV and with this norm, BV is a Banach space (see Aumann and Shapley [1974, Corollary

4.2, and Proposition 4.3]. The following notation is standard in the literature:

$$\begin{aligned}
\mathrm{NA} &= \text{set of non-atomic measures on } (I, \mathcal{B}_I)\\
\mathrm{NA}^1 &= \text{set of probability measures in NA}\\
\mathrm{pNA} &= \|.\|_{BV} - \text{closure of linear space spanned by powers of } \mu \in \mathrm{NA}\\
\mathrm{bv}'\mathrm{NA} &= \|.\|_{BV} - \text{closure of linear space spanned by } f \circ \mu, \text{ where } f: I \to \Re\\
&\quad \text{is of bounded variation, continuous at 0 and 1, and } f(0) = 0, \text{and}\\
&\quad \mu \text{ is a non-negative, non-atomic probability measure on } (I, \mathcal{B}_I).
\end{aligned}$$

It is known that FA, NA and pNA are all closed subspaces of BV.

2.1 Generation of random orders. Two main features of the random order approach to values of games with finite sets of players are that (1) each automorphism [2] generates a distinct ordering of players, i.e., the set of orders is the same as the group of automorphisms; (2) for all games, the mathematical expectation of the random marginal contribution set function is symmetric with respect to the group of automorphisms if and only if each random ordering of players is equally likely (see Raut [1993]). In the finite players case, the main reason why the expected marginal contribution set function becomes symmetric for any game and with respect to the full group of permutations is that every player is equally likely to form a coalition with a set of players of any size and names in a random order. We want to adopt these two features to the continuum case.

Each $\theta \in \Theta$ generates a binary relation, $\succ_\theta \subset I \times I$ defined by

$$\text{for any } s, t \in I,\ s \succ_\theta t \Leftrightarrow \theta(s) > \theta(t).$$

Recall that an order $\succ$ on a set X is a *linear order*, which is also known as *total order* if for any $x, y \in X$, $x \neq y$, either $x \succ y$ or $y \succ x$, for no $x \in I$, $x \succ x$, and for any $x, y, z \in I$, $x \succ y$, $y \succ z \Rightarrow x \succ z$. A total order is a particular type of preference order. We will refer to a total order in this paper simply as an *order*. It is easy to verify that the binary relation $\succ_\theta$ generated by an automorphism θ is an order on I and that each $\theta \in \Theta$ generates a distinct order. Let $\bar{I} = I \cup \{\infty\}$, and for all $\theta \in \mathcal{G}$, we assume that $\theta(\infty) = \infty$. For an order $\succ_\theta$, $\theta \in \mathcal{G}$, and a $s \in \bar{I}$, define *an initial segment* $I(s, \theta)$ by $I(s, \theta) = \{t \in I \mid \theta(s) > \theta(t)\}$. We view $I(s, \theta)$ as the set of players who are before player s in the order $\succ_\theta$.

In the continuum case two Borel automorphisms of I may generate the same ordering of I. For instance, $\theta \in \mathcal{G}$, defined by $\theta(x) = x^2$, $x \in I$ and the identity element $e \in \mathcal{G}$, defined by $e(x) = x, x \in I$, both generate the standard order $\succ_e$. Thus the set of orderings of players and the group of Borel automorphisms of players are not the same set. We derive the set of orders Ω that are generated by a given group of automorphisms Θ as follows:

Define an equivalence relation $\sim$ on $\Theta \times \Theta$ by,

$$\theta_1 \sim \theta_2, \text{ for } \theta_1, \theta_2 \in \Theta \Leftrightarrow \theta_1, \theta_2 \text{ generate the same order on } I.$$

Let $\Theta_e = \{\theta \in \Theta | \theta \sim e\}$. It can be easily shown that Θ_e is a subgroup of Θ and the set of distinct orders, Ω, generated by the automorphisms in Θ is the set of right cossets given by

$$\Omega \equiv \Theta / \Theta_e \equiv \{\Theta_e \theta | \theta \in \Theta\}$$

In the finite player case, the set of automorphisms of players is finite and for finite sets the concept of equal likelihood is intuitive. In the continuum case,

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[2] In the finite players case an automorphism is known as permutation.

however, the set of automorphisms of the players is uncountable. An analogue of the equal likelihood in the continuum set is the following concept of an invariant measurable group or invariant measure. This concept requires the underlying space to have a group structure:

Definition 1 A measure space $(\Theta, \mathcal{A}, \Gamma)$ is said to be an *invariant measurable group* if Θ is a group, the map $(\theta_1, \theta_2) \to \theta_1\theta_2^{-1}$ from $(\Theta \times \Theta, \mathcal{A} \times \mathcal{A})$ onto $(\Theta, \mathcal{A})$ is measurable, and Γ is $\sigma-$finite, not identically zero, and right invariant, i.e., $\Gamma(E\theta) = \Gamma(E)$, for all $E \in \mathcal{A}$, and $\theta \in \Theta$, where $E\theta \equiv \{\sigma\theta | \sigma \in E\}$. Γ is known as *right invariant measure.*[3] Furthermore, when Γ is a probability measure, a measurable group $(\Theta, \mathcal{A}, \Gamma)$ is said to be a *right invariant probability measurable group.*

In general Θ_e is not a normal[4] subgroup of Θ and hence Ω is not necessarily a group. To see this, let $\theta \in \mathcal{G}$ and $\theta_e \in \mathcal{G}_e^*$ be defined by

$$
\begin{aligned}
\theta(x) &= \begin{cases} 1-x & \text{if } 0 \le x < 1/2 \\ x - 1/2 & \text{if } 1/2 \le x \le 1. \end{cases} \\
\theta_e(x) &= \begin{cases} .01x & if\ 0 \le x < .8 \\ .008 + 4.96(x - .8) & if\ .8 \le x \le 1 \end{cases}
\end{aligned}
$$

Let $t = .4$ and $s = .3$. Thus $\theta_e(s) < \theta_e(t)$, but $(\theta^{-1}\theta_e\theta)(s) = .507 > .506 = (\theta^{-1}\theta_e\theta)(t)$, thus $\theta^{-1}\theta_e\theta \notin \mathcal{G}_e$.

Thus we do not have the group structure on Ω that we needed to extend the concept of equal likelihood of orderings in Ω. But Ω is a homogeneous space acted on by the group Θ, and for homogeneous spaces there is a natural concept of invariant measure (see Parthasarathy [1977, Section 55]; or Segal and Kunz [1978, Section 7.4]). In our set-up, however, we can use the *natural map* $\Pi : \Theta \to \Omega$ defined by $\Pi(\theta) = \Theta_e\theta$ to induce an invariant probability measure structure $(\Omega, \mathcal{B}, \mu)$ on the homogeneous space Ω of induced orders. $(\Omega, \mathcal{B}, \mu)$ will be referred to as a *set of random orders.*

2.2 Θ-symmetric random order value operator. Given a game V, and an order $\succ_\theta$, $\theta \in \mathcal{G}$, a *marginal contribution set function,* $\phi^\theta V$ on $(I, \mathcal{B}_I)$ is a measure on $(I, \mathcal{B}_I)$ such that

$$
\left(\phi^\theta V\right)(I(s,\theta)) = V(I(s,\theta)),\ \forall\ s \in \bar{I} \tag{2.1}
$$

Notice that for any $\theta, \theta' \in \Theta$, such that $\theta \sim \theta'$, we have $I(s,\theta) = I(s,\theta')$; hence it follows from (2.1) that $\phi^\theta V(S) = \phi^{\theta'} V(S)$ for all $S \in \mathcal{B}_I$. This allows us to unambiguously define $(\phi^\omega V)(S) = (\phi^\theta V)(S)$ where θ is such that $\omega = \Theta_e\theta$.

The *expected marginal contribution set function for a game V* is a set function $\Phi_\Gamma V$ defined by

$$
\begin{aligned}
(\Phi_\Gamma V)(S) &= \int_\Omega (\phi^\omega V)(S) d\mu(\omega) \\
&= \int_\Theta (\phi^\theta V)(S) d\Gamma(\theta),\ S \in \mathcal{B}_I.
\end{aligned} \tag{2.2}
$$

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[3]When Θ is a locally compact topological group, and $\mathcal{A}$ is the Borel σ-algebra, such that $\Gamma(U) > 0$, for every non-empty open set $U \subset \Theta$, then the Borel measure Γ is known as Haar Measure.

[4]N is a normal subgroup of G if for all $\theta \in$ G, we have $\theta^{-1}\nu\theta \in$ N for all $\nu \in$ N.

The second equality follows from the change of variable formula for Lebesgue integrals and the facts in the previous paragraph. Let us define the space of games:

$$L1(\Theta, \Gamma) = \left\{ V \in G_I \mid \phi^\theta V(S) \text{ in (2.2) is integrable for all } S \in \mathcal{B}_I \right\}. \qquad (2.3)$$

For the operator Φ_Γ defined in (2.2) to yield a random order value operator in the continuum case, it should have three basic properties (which are shown to hold in Raut [1995]): **First**, for any game V and any order $\succ_\theta$, $\theta \in \Theta$, if there exists a measure $\phi^\theta V$ satisfying (2.1), it should be unique so that for each $S \in \mathcal{B}_I$, $\phi^\theta V(S)$ is a function of θ. **Second**, in order for the operator Φ_Γ to be Θ-symmetric with respect to a given subgroup of automorphisms, $L1(\Theta, \Gamma)$ defined in (2.3) must be a Θ-symmetric linear subspace of G_I. **Third**, the approach is of little use if for a given symmetry group of automorphisms, Θ, two different probability measurable structures assign two different finitely additive set functions to a game. The second part of Raut [1995, Theorem 1] shows that the mathematical expectation in (2.2) depends only on the fixed subgroup of automorphisms, Θ, but not on a specific invariant probability measurable group structure on Θ.

For a given fixed subgroup of automorphisms $\Theta \subset \mathcal{G}$ and a Θ-symmetric linear subspace of games $Q \subset G_I$, the operator $\Phi_\Gamma : Q \to FA$ defined in (2.2) with respect to an invariant probability measurable group structure $(\Theta, \mathcal{A}, \Gamma)$ on Θ such that $Q \subset L1(\Theta, \Gamma)$ is said to be a *Θ-symmetric random order value operator* on Q. Raut [1995, Theorem 1] assures that when such an operator exists, it is independent of a specific invariant probability measurable group structure on Θ and it coincides with the Θ-symmetric axiomatic value operator on Q. In fact, a Θ-symmetric random order value operator is a particular characterization of the Θ-symmetric axiomatic value operator. In particular, therefore, if we take Θ to be the full automorphism group $\mathcal{G}$, then the existence of an Aumann-Shapley axiomatic value operator on Q can be reduced to the question of the existence of an invariant probability measurable group structure, $(\mathcal{G}, \mathcal{B}, \Gamma)$, with the property that $Q \subset L1(\mathcal{G}, \Gamma)$.

In our random order approach, the set of orders induced by $\mathcal{G}$ is a proper subset of the set of measurable orders considered by Aumann and Shapley in their random order approach. Therefore we may recast the Aumann-Shapley impossibility issue and expect a possibility result in our framework:

Question 1: *Could there exist an invariant probability measurable group structure on $\mathcal{G}$, such that* pNA $\subset L1(\mathcal{G}, \Gamma)$?

Indeed, on any group Θ, there always exists a right invariant probability measurable group structure, for instance, the trivial, coarsest σ-algebra, $\mathcal{B} = \{\emptyset, \Theta\}$ with a trivial probability measure that assigns 0 to empty set and 1 to the whole set. The coarser the σ-algebra, the more meager are the sets of measurable and integrable functions, and hence fewer games belong to $L1(\Theta, \Gamma)$ which may not include games in pNA. If the answer to the above question is still negative, we may find a positive answer when we restrict the question to other possibly smaller classes of economically important games than pNA. I conjecture the proof of Aumann-Shapley impossibility theorem could be adopted to the present framework to produce a negative answer to the above question. Very little is known about the structure of group $\mathcal{G}$ that can shed light on the above issues, and I have not pursued these issues any further in this paper either. Instead, I have carried my analysis along the following line:

Question 2: *If we restrict the symmetry of the value operator to a "reasonable" proper subgroup of automorphisms Θ, could we construct a probability measurable* *group structure*, $(\Theta, \mathcal{A}, \Gamma)$ *on* Θ *such that* $L1(\Theta, \Gamma)$ *contains a large set of games including* pNA *and on a large class of economically important games from* pNA, the Θ-*symmetric random order value operator coincides with the Aumann-Shapley axiomatic value operator?*

We carried out the above type of inquiry for the $\breve{\Theta}$-symmetric random order value operator $\Phi_{\breve{\Gamma}}$ for the uncountably large subgroup $\breve{\Theta}$ of the group of l.m.p. automorphisms with the invariant probability measurable group structure, $\left(\breve{\Theta}, \breve{\mathcal{B}}, \breve{\Gamma}\right)$. This was introduced in Raut [1997], and we explain its structure in Section 3 .

In the context of restricting the symmetry group, two types of issues are addressed in the axiomatic approach to value. **First**, The largest subspace of BV on which Aumann-Shapley established the existence and uniqueness of $\mathcal{G}$-symmetric axiomatic value operator is bv$'$NA, and they also showed that there cannot exist a $\mathcal{G}$-symmetric axiomatic value operator on all of BV. Ruckle [1982] addressed the question: If the symmetry of the value operator is restricted to a proper subgroup, could there exist an axiomatic value operator on all of BV? Ruckle has shown that if Θ is taken to be any "locally finite" group then there exists a Θ-symmetric axiomatic value operator on all of BV. A group is said to be *locally finite* if for any finite number of elements from the group there is a finite subgroup which contains these elements. Monderer and Ruckle [1990] gave further refinement of this result. We have an analogous result. To describe it we need the following definition:

Definition 2 A set function $V \in G_I$ is said to be *normalized* if (i) $V(A_n) \to 0$ as $n \to \infty$ for any sequence of sets, $A_n \in \mathcal{B}_I$, $A_n \downarrow \emptyset$ as $n \to \infty$, and (ii) $V(A_n) \to V(A)$ as $n \to \infty$ for any sequence of sets, $A_n \in \mathcal{B}_I$, $A_n \uparrow A$ as $n \to \infty$, where $A \in \mathcal{B}_I$.

Let us denote by NBV = the set of normalized set functions from BV. It can be easily seen that NBV is a linear space. The result is as follows:

Theorem 1 *There exists a unique* $\breve{\Theta}$-*symmetric random order value* $\Phi_{\breve{\Gamma}}$ *on* NBV.

Second, Monderer [1986, 1989] addressed the question: If Θ is a proper subgroup of $\mathcal{G}$, then on which linear subspace of pNA, does there exist a unique Θ -symmetric value operator? Or equivalently, on which subspace of games in pNA, Θ-symmetric value operator coincides with the Aumann-Shapley axiomatic value operator? He has shown that when Θ is taken to be the group of μ-measure ($\mu \in NA$) preserving automorphisms and the linear space of games restricted to pNA(μ) (the closed subspace of games in pNA spanned by polynomials of non-atomic probability measures which are absolutely continuous with respect to the given non-atomic probability measure μ).[5] Then there exists a unique Θ-symmetric axiomatic value operator satisfying dummy axiom on pNA(μ). With a little modification in the proof in Raut [1995, Theorem 3]) we have the following analogous result for our random order approach.

Theorem 2 *Let* $f : I \to \Re$ *be absolutely continuous, and* $\lambda \in NA^1$, *then the unique* $\breve{\Theta}$-*symmetric random order value of the scalar measure game* $f \circ \lambda$ *yields*

[5] See his economic justifications for restricting the symmetry group and the class of games in this particular way so that his theory applies to a large class of market games.

the following diagonal formula:

$$\Phi_{\breve{\Gamma}}[f \circ \lambda](S) = \lambda(S) \int_{\bullet}^{1} f'(t) d\lambda.(t) \tag{2.4}$$

Thus, the $\breve{\Theta}$-symmetric random order value operator $\Phi_{\breve{\Gamma}}$ coincides with the Aumann-Shapley axiomatic value operator on all of lpNA(λ).

With symmetry and generation of random orders restricted to a subgroup of μ-measure preserving automorphisms $\breve{\Theta}(\mu)$,[6] it is easy to modify our analysis so that the above result is true for lpNA(μ). I conjecture that the above result holds for pNA(μ). At this point, I have not been able to prove it.

3 On the symmetry subgroup $\breve{\Theta}$

The essence of the success of the random order approach to characterize the Shapley value of games with finite set of players ask the following question: *Does there exist an uncountably large subgroup of automorphisms in $\mathcal{G}$ that can be equipped with an invariant probability measurable group structure, $(\bullet, \mathcal{A}, \Gamma)$, such that in a randomly selected order every player is equally likely to have placed before it a set of players of any size containing players from anywhere in the unit interval?* This notion beckons to the strongly mixing automorphisms. To fix ideas, let us consider mixing with respect to Lebesgue measure λ. A Lebesgue measure preserving automorphism, $\theta \in \mathcal{G}$, is said to be *strongly mixing* if $\lim_{n\to\infty} \lambda(\theta^{-n}E \cap F) = \lambda(E \cap F)$, for all $E, F \in \mathcal{B}_I$. In essence a strongly mixing automorphism, θ, allows thorough mixing of any set of players $t \in E$ with any other player in the unit interval I by producing an orbit $\mathcal{Q}(E) = \{\theta^n t | t \in E, n = 0, 1, ..\}$ which is dense and uniformly spread allover I. A l.m.p. automorphism is *weakly mixing* if $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \lambda\left(\theta^{-j}E \cap F\right) = \lambda(E \cap F)$. It is known that with respect to the "weak topology", the set of such automorphisms is of the first category and the set of weakly mixing automorphisms is of the second category. This means that generically a measure preserving automorphism is a weakly mixing but not strongly mixing. Aumann [1967] [7] has shown that it is impossible to find an invariant probability measurable group structure on the whole group of l.m.p. automorphisms which satisfy further conditions that the real valued function $f(\theta) \equiv \lambda(E \cap \theta F)$ is measurable for all $E, F \in \mathcal{B}_I$.

Thus in our approach, we seek to achieve thorough mixing of players with the help of actions of an uncountably large subgroup $\breve{\Theta}$ of l.m.p. automorphisms. This we obtain as a (projective) limit of an increasing sequence of "carefully constructed" increasing finite subgroups, Θ_n, $n \geq 0$ of l.m.p. automorphisms. We describe the construction of $\breve{\Theta}$ briefly in this section, (for details, see Raut [1997]). It is, however, interesting to note that in our construction we get thorough mixing with the help of automorphisms in Θ_n's which are all recurrent of period n.

We also want to equip $\breve{\Theta}$ with a fine enough measurable group structure $\left(\breve{\Theta}, \breve{\mathcal{B}}, \breve{\Gamma}\right)$ so that it allows a sufficiently rich set of games in $L1(\breve{\Theta}, \breve{\Gamma})$. We impose the following separability requirement on our measurable group structure:

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[6]This subgroup is, in fact, a translation of the group $\breve{\Theta}$ with an automorphism $\xi \in \mathcal{G}$, where ξ is such that $\mu = \lambda\xi^{-1}$. That there exists such an automorphism ξ can be found in Parthasarathy [1977].

[7]I am grateful to Professor Robert Aumann for drawing my attention to this result.

Definition 3 A measurable group $(\Theta, \mathcal{A}, \Gamma)$ is *separated* [8] if $\forall \theta \in \Theta$, $\theta \neq e$, there exists $E \in \mathcal{A}$ such that $0 < \Gamma(E) < \infty$ and $\Gamma(E\theta\Delta E) > 0$, where Δ is the symmetric difference operator between two sets.

We define the increasing sequence of finite groups, Θ_n, $n \geq 0$ each containing l.m.p. automorphisms that are discontinuous at only a finite number of points of I. In the n-th automorphism group, Θ_n, the discontinuities of the automorphisms are at the points $\frac{k}{2^n}, k = 1, ..., 2^n - 1$. These $2^n - 1$ points in I determine 2^n dyadic subintervals of I: $I_k = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$, $k = 0, 1, ..., 2^n - 1$. In order for the automorphisms to be Lebesgue measure preserving, we assume that in each subinterval I_k, the automorphisms are linear with slope $\pm$ 1.

Let $N_n = \{0, 1, 2, ..., 2^n - 1\}$. We represent each automorphism in Θ_n by a pair of functions, π_n and $\mathcal{O}_n$ such that $\pi_n : N_n \longrightarrow N_n$ is a permutation of N_n and $\mathcal{O}_n : N_n \longrightarrow \{-1, 1\}$ is a map as follows: For each $k \in N_n$, $\pi_n(k)$ specifies which subinterval of the unit interval the image of the k^{th} subinterval be mapped to, and $\mathcal{O}_n \circ \pi_n(k)$ specifies the slope of the automorphism that the image subinterval will take. Sometimes we will refer to $\mathcal{O}_n$ as the *slope map*. We denote such an automorphism as described above by the symbol

$$\theta_n = (\pi_n(k), \mathcal{O}_n \circ \pi_n(k))_{k=0}^{2^n-1} . \tag{3.1}$$

An equivalent description of the above automorphism that we will often use is the following:

$$\theta_n(x) = \begin{cases} \frac{\pi_n(k)}{2^n} - \frac{k}{2^n} + x & \text{if } x \in I_k \text{ and } \mathcal{O}_n(\pi_n(k)) = +1 \\ \frac{\pi_n(k)+1}{2^n} - \frac{k}{2^n} - x & \text{if } x \in I_k \text{ and } \mathcal{O}_n(\pi_n(k)) = -1 \\ & k = 0, 1, ...2^n - 1 \end{cases} \tag{3.2}$$

We use θ_n to mean the representation (3.1) and $\theta_n(.)$ or $\theta_n(x)$ to mean the representation (3.2) of an element in Θ_n.

Let $\Im = \{+1, -1\}$ denote the set of slopes. With the usual multiplication operation of real numbers and with $+1$ as the identity element, it is trivial to show that $\Im$ is a group.

Let

$$\Theta_n = \left\{ \begin{array}{c} \theta_n : I \rightarrow I, \text{ defined by (3.2)} \mid \pi_n \text{ is a permutation of } N_n \\ \text{and } \mathcal{O}_n \text{ is an orientation of the } 2^n \text{sub-intervals.} \end{array} \right\}$$

There are $(2^{2^n}.2^n!)$ total number of elements in Θ_n. It can be easily seen that Θ_n is a subgroup of l.m.p. automorphisms. For instance, when $n = 0$, there is no subdivision of I, and thus

$$\Theta_0 = \{\theta_0 = (\pi_0(0), \mathcal{O}_0(0)) \mid \mathcal{O}_0(0) \in \Im\} .$$

Note that Θ_0 has only two elements.

For further illustration of the above concepts, in panel (a) of Figure 1, we have shown the graph of $\theta_2(x)$ corresponding to the permutation, $\pi_2(1) = 2$, $\pi_2(2) = 4$, $\pi_2(3) = 1$ and $\pi_2(4) = 3$, and the orientation, $\mathcal{O}_2(1) = +1$, $\mathcal{O}_2(2) = -1$, $\mathcal{O}_2(3) = +1$, and $\mathcal{O}_2(4) = -1$.

[8] This separation notion for measurable groups is the analogue of Hausdorff separation axiom for topological spaces, see Halmos [1950, pp. 273].

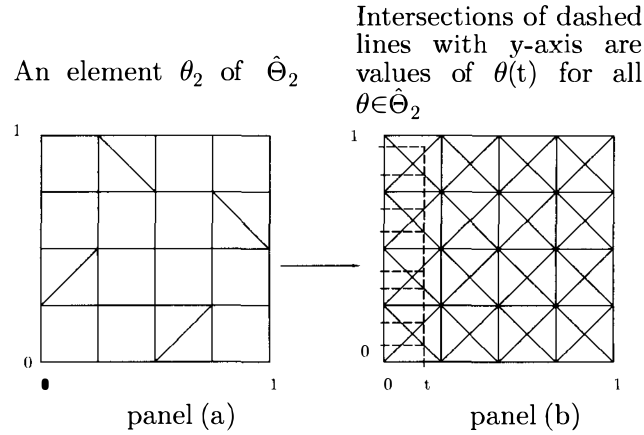

Figure 1

Let us examine the kind of randomization of the players that are performed by the random orders in Θ_n for large n. For illustration purpose, let us consider the automorphism $\theta_2 \in \Theta_2$ that is depicted in panel (a) of Figure 1. Note that the set of players that are placed before player t, $t \in I$, in the random order $\theta_2 \in \ \Theta_2$ is given by

$$I(\{t\},\theta_2) = \begin{cases} [0,t) \cup I_3 & \text{if } \ t \in I_1 \\ \\ I_1 \cup (t, \frac{1}{2}) \cup I_3 \cup I_4 & \text{if } \ t \in I_2 \\ \\ [\frac{1}{2}, t) & \text{if } \ t \in I_3 \\ \\ I_1 \cup I_3 \cup (t, 1] & \text{if } \ t \in I_4 \end{cases}$$

Note that the nature of randomization produced by an element of Θ_n depends on the permutation π and the orientation $\mathcal{O}$. Let us fix a $t \in I$ and suppose $t \in I_1$. Consider the initial segments of player t in each of the random orders $\theta_n \in \Theta_n$, that has the same value for t, say $\theta_n(t) = t_0$. All of these random orders will have either positive orientation or negative orientation. Let us assume that they have positive orientation. Let us denote by [x], $x \in \Re$, as the greatest integer in x. The way the initial segments are randomized by these random orders is that $[\frac{2^n}{t_0}]$ sub-intervals from the set of all sub-intervals except I_1 are randomly selected and then placed before the set of points [0,t) in all possible permutations. For very large n, size of each sub-intervals is very small, and hence for large n, all these $\theta_n^{\prime}s$ with fixed value of $\theta_n(t) = t_0$ are placing almost any infinitesimally small subintervals of I that can fit in an interval of size $[0, t_0]$. The size of the interval also varies as we vary t_0 in the orbit $\mathcal{Q}_n = \{\theta(t_0) \mid \theta \in \Theta_n\}$.

In panel (b) of Figure (1), we have graphed all the elements of Θ_2. The set $\mathcal{Q}_2$ is shown as the intersection of the dash lines with the y-axis. It is trivial to note that $\Theta_n \subset \Theta_{n+1}$ and as $n \to \infty$, the number of elements in $\mathcal{Q}_n$ becomes large and are spread uniformly over I.

It can be easily shown that the limiting set, $\Theta \equiv \cup_{n=1}^{\infty}\Theta_n = \lim_{n\to\infty}\Theta_n$ is a group of l.m.p. automorphisms, but contains only countable number of elements, and thus cannot admit an invariant probability measurable structure. It is interesting to note, however, that Θ is locally finite. Thus we have an example of a locally finite symmetry group Θ such that there does not exist a Θ-symmetric random order value operator on any space of games at all.

We can, however, construct a large subgroup of l.m.p. automorphisms with a different kind of limit, namely, the projective limit of the above sequence of Θ_n limiting further the elements in each of them as follows: We use $\hat{\Theta}_n$ to denote the modified Θ_n. $\hat{\Theta}_n, n \geq 0$ are recursively constructed as follows:

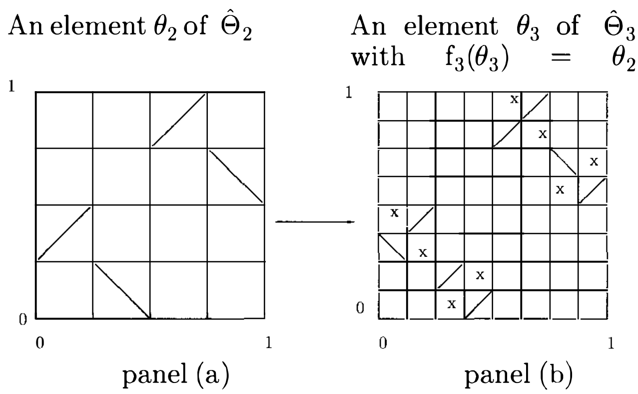

Figure 2

For $n = 0$, take $\hat{\Theta}_0 = \Theta_0$. To define $\hat{\Theta}_1$, note that we have two dyadic sub-intervals of I denoted as I_0 and I_1. Each $\theta_0 \in \hat{\Theta}_0$, induces a unique permutation π_{1,θ_0} of $N_1 = \{0, 1\}$ defined by

$$\pi_{1,\theta_0}(j) = i \text{ if for all } x \in I_j,\ \theta_0(x) \in I_i,\ i, j \in N_1.$$

Given $\theta_0 \in \hat{\Theta}_0$, let us denote by

$$A_1(\theta_0) = \left\{\theta_1 = (\pi_{1,\theta_0}(k), \mathcal{O}_1(\pi_{1,\theta_0})(k))_{k=0}^{1} \mid \mathcal{O}_1(\pi_{1,\theta_0})(j) \in \Im, j = 0, 1\right\}.$$

We now define $\hat{\Theta}_1$ by

$$\hat{\Theta}_1 = \bigcup_{\theta_0 \in \hat{\Theta}_0} A_1(\theta_0).$$

Note that each $A_1(\theta_0)$ has $2 \times 2 = 4$ elements and hence $\hat{\Theta}_1$ has $2 \times 4 = 8$ elements.

Let us suppose that we have already defined $\hat{\Theta}_{n-1}$. We now define $\hat{\Theta}_n$ from $\hat{\Theta}_{n-1}$.

Let the 2^n dyadic sub-intervals at stage n be denoted as $I_0, \ldots\ I_{2^n-1}$. Each $\theta_{n-1} \in \hat{\Theta}_{n-1}$ induces a unique permutation $\pi_{n,\theta_{n-1}}$ of the set N_n defined by

$$\pi_{n,\theta_{n-1}}(j) = i \text{ if for all } x \in I_j,\ \theta_{n-1}(x) \in I_i,\ i, j \in N_n. \tag{3.3}$$

For each $\theta_{n-1} \in \hat{\Theta}_{n-1}$, we define

$$A_n(\theta_{n-1}) = \left\{ \begin{array}{c} \theta_n = \left(\pi_{n,\theta_{n-1}}(k), \mathcal{O}_n(\pi_{n,\theta_{n-1}})(k)\right)_{k=0}^{2^n-1} \mid \\ \mathcal{O}_n(\pi_{n,\theta_{n-1}})(i) \in \Im \ \forall \ i \in N_n \end{array} \right\}$$

and

$$\hat{\Theta}_n = \bigcup_{\theta_{n-1} \in \hat{\Theta}_{n-1}} A_n(\theta_{n-1}).$$

For each $n \geq 1$, we define the multiplication operation in $\hat{\Theta}_n$ as the composition of functions, namely, for $\hat{\theta}_n$, $\hat{\theta}'_n \in \hat{\Theta}_n$, we define $\hat{\theta}_n \hat{\theta}'_n = \hat{\theta}_n\left(\hat{\theta}'_n(x)\right)$, $x \in I$, and for each $n \geq 1$, we define the projection maps $f_n : \hat{\Theta}_n \to \hat{\Theta}_{n-1}$, $f_n(\theta_n) = \theta_{n-1}$, where θ_{n-1} is such that $\theta_n \in A_n(\theta_{n-1})$.

To get an idea about these projection maps, in panel (b) of Figure 2, we have drawn a $\theta_3 \in \hat{\Theta}_3$ whose projection using the map f_3 is $\theta_2 \in \hat{\Theta}_2$.

Denote by

$$\breve{\Theta} = \left\{ \breve{\theta} = (\theta_0, \theta_1, \theta_2, ...) \mid \theta_n \in \hat{\Theta}_n, \forall n \geq 0 \text{ and } f_n(\theta_n) = \theta_{n-1}, \forall n \geq 1 \right\}.$$

For any two elements $\breve{\theta} = (\theta_0, \theta_1, \theta_2, ...)$ and $\breve{\theta}' = (\theta'_0, \theta'_1, \theta'_2, ...)$ from $\breve{\Theta}$, define the multiplication operation $\breve{\theta} \circ \breve{\theta}'$ by

$$\breve{\theta} \circ \breve{\theta}' = (\theta_0\theta'_0, \theta_1\theta'_1, \theta_2\theta'_2, ...)$$

By Raut [1997, Proposition 3], $f_n(\theta_n\theta'_n) = f_n(\theta_n)f_n(\theta'_n) \in \hat{\Theta}_{n-1}$. Hence, $\breve{\theta} \circ \breve{\theta}' \in \breve{\Theta}$. With $\breve{\theta}^{-1} = (\theta_0^{-1}, \theta_1^{-1}, \theta_2^{-1}, ...)$ as the inverse of $\breve{\theta} = (\theta_0, \theta_1, \theta_2, ...)$, and with $\breve{e} = (e_0, e_1, ...)$, where e_n is the identity element of $\hat{\Theta}_n$ as the unit element, we note that $\breve{\Theta}$ is a group.

Define for $n \geq 0$ the projection maps $\pi_n : \breve{\Theta} \to \hat{\Theta}_n$ by

$$\pi_n(\breve{\theta}) = \theta_n, \quad \text{where} \quad \breve{\theta} = (\theta_0, \theta_1, \theta_2, ...).$$

Let $\mathcal{F} = \cup_{n=1}^{\infty} \pi_n^{-1}(\mathcal{B}_n)$. It can be easily shown that $\mathcal{F}$ is a Boolean algebra. Let $\breve{\mathcal{B}}$ be the $\sigma-$algebra generated by $\mathcal{F}$. $\left(\breve{\Theta}, \breve{\mathcal{B}}\right)$ is called the *projective limit* of the sequence of measure spaces, $(\hat{\Theta}_n \mathcal{B}_n)$, $n \geq 0$ through the maps $f_n, n \geq 1$.

The following theorem summarizes the main results from Raut [1995, 1997] for the projective limit group $\left(\breve{\Theta}, \breve{\mathcal{B}}, \breve{\Gamma}\right)$:

Theorem 3 *There exists a unique right invariant probability measure, $\breve{\Gamma}$ on the projective limit, $(\breve{\Theta}, \breve{\mathcal{B}})$ of the sequence of measurable groups, $\left(\hat{\Theta}_n, \mathcal{B}_n, \Gamma_n\right)_0^{\infty}$, through the sequence of homomorphisms, $\{f_n\}_0^{\infty}$, such that*

(i) $\breve{\Gamma}\pi_n^{-1} = \Gamma_n$

(ii) $\left(\breve{\Theta}, \breve{\mathcal{B}}, \breve{\Gamma}\right)$ *is an uncountably large separated probability measurable group.*

(iii) *For each $\breve{\theta} = (\theta_0, \theta_1, ..., \theta_n, ...) \in \breve{\Theta}$, the limit $\breve{\theta}(t) = \lim_{n\to\infty} \theta_n(t)$ exists for all $t \in I$ and the limit function $\theta : I \to I$ is a Lebesgue measure preserving automorphism.*

(iv) *The projective limit group $\left(\breve{\Theta}, \breve{\mathcal{B}}, \breve{\Gamma}\right)$ is isomorphic to the unit interval with Lebesgue measure, $(I, \mathcal{B}_I, \lambda)$.*

Let us examine the limitations that are imposed on the pattern of randomization by the projective limit group as compared to the limit group Θ that we discussed earlier. Let us fix a $t \in I$ and suppose $t \in I_1$. Consider the initial segments of player t in each of the random orders $\theta_n \in \breve{\Theta}_n$, that has the same value for t, say $\theta_n(t) = t_0$. From Figure 2 it is clear that all these random orders place a particular type of sets of Lebesgue measure t_0 before player t; and the type of the sets depends on t_0, and the order of the dyadic subdivision, n. For example, suppose $t_0 < 1/2$, then all these random orders do not place any players from the interval [1/2,1]. Thus for a given size t_0, $0 \leq t_0 \leq 1$, the random orders in $\breve{\Theta}$ allow t to form coalition only with certain sets of players of size t_0 but not every (Borel) set of players whose size is t_0. On the other hand, t gets to have any given player placed before him in a suitable random order and a suitable size t_0.

For games in which the worth of a coalition depends only through its size but not through any of their other identities, i.e., for anonymous games for instance such as the games in lpNA(λ), we expect and we have shown formally in Raut [1995] that the expected marginal contribution of a player with respect to the group of random orders, $\left(\breve{\Theta}, \breve{\mathcal{B}}, \breve{\Gamma}\right)$, coincides with the axiomatic value for these games as characterized by Aumann and Shapley [1974]. See Theorem 2 for details.

4 Final remarks

Remark 1 There are economically important non-smooth games which neither belong to bv′NA, MIX, nor even to ASYMP. Mertens [1988] extended the diagonal formula for value to a very powerful closed subspace of games in BV, known as *Mertens space*, on which the extended diagonal formula provides a value operator of norm 1 and the Mertens space was shown to include all well known spaces such as bv′NA, ASYMP, DIFF and DIAG. Both J. F. Mertens and Abraham Neyman suggested to me to examine if Mertens space belongs to $L1(\breve{\Theta}, \breve{\Gamma})$. I have not tried to get a general answer to this question, instead I show that the $\breve{\Theta}$-symmetric random order value exists for the non-smooth game of "n-handed gloves markets" considered in example 19.2 of Aumann and Shapley [1974, p.136]: $V(S) = \min\{\mu_1(S), \mu_2(S), ..., \mu_n(S)\}$, $\mu_i \in NA^1$, $i = 1, 2, ...n$, and $S \in \mathcal{B}_I$. This kind of non-smooth games arise in economies with strong complementarities. Aumann and Shapley showed that this game did not belong even to ASYMP when $n > 2$. One of the motivations for Mertens [1988] to extend the diagonal formula to the Mertens space was to include such games in the space. Notice that V is of bounded variation. Since each μ_i is a non-atomic probability measure, the game $V(S)$ is normalized and hence belongs to NBV. Thus there exists a unique $\breve{\Theta}$-symmetric random order value for V.

Remark 2 An important issue regarding our reformulated random order approach is: What characteristics of the group $\breve{\Theta}$ that makes the random order value coincides with the axiomatic value on lpNA(λ)? In Section 3, I argued that a random order generated according to the probability model $\left(\breve{\Theta}, \breve{\mathcal{B}}, \breve{\Gamma}\right)$ has the characteristics that the random set of players that is placed before any given player is equally likely to be of any size $s \in [0, 1]$; the anonymous games in which worth of a coalition depends only through its size not names such as games in lpNA(λ), each player gets the average of the set of all possible marginal contributions and thus average is fully symmetrized in the sense that the value thus obtained is symmetric with respect to the full group of automorphisms. Locally finite groups of automorphisms may not do the job, as we have illustrated in Section 3. The games that arise in most economic applications are anonymous. However, for a wider applicability of the present approach, we must construct a larger invariant probability measurable group structure $(\Theta, \mathcal{B}, \Gamma)$ than $\left(\breve{\Theta}, \breve{\mathcal{B}}, \breve{\Gamma}\right)$, so that the random order value $\Phi_\Gamma(V)$ with respect to it also fully symmetrizes non-anonymous games.

Remark 3 Robert Aumann pointed out to me that for an alternative reformulation of random order approach to value, one might give up the measure theoretic model of the player set, i.e., $(I, \mathcal{B}_I)$, and instead consider the player space to be torus or other topological spaces with more well-behaved automorphism groups. It should be noted that there can exist only two orders on any topological space that is connected. This, for instance, will greatly simplify our analysis of random order value. We do not know, however, what kind of fairness such a symmetry group entails and what kind of economic situations are appropriate for such models; most of the economic models with a continuum of agents, however, have employed measure theoretic structures for the space of agents, and thus we must begin to imagine the nature and study the implications of economic models with a topological space of agents.

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