Subgame perfect manipulation of children by overlapping generations of agents with two-sided altruism and endogenous fertility[1]

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Abstract

In this paper we consider an overlapping generations model with endogenous fertility and two-sided altruism and show the limitations of applying commonly used open loop Nash equilibrium in characterizing equilibrium transfers from parents to children in the form of bequest, and transfers from children to parents as voluntary old-age support. Since in our model children are concerned with parents' old-age consumption, agents have incentives to save less for old age and to have more children so as to strategically induce their children to transfer more old-age support. We formulate such strategic behavior within a sequential multi-stage game and use the notion of subgame perfect equilibrium to study the consequences of such strategic manipulations on private intergenerational transfers, fertility and savings decisions, and on Pareto optimality of equilibrium allocation. We then examine the role of social security to correct such strategic distortions.

Keywords: *two-sided altruism, endogenous fertility, subgame perfect manipulation of children, social security.*

1 Introduction

In standard pure exchange overlapping generations (OLG) economies agents have life-cycle utility function. These models do not explain private intergenerational transfers within family and have no bearings on the effects of public transfers policies such as social security on private intergenerational transfers, savings and fer-

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tility. Moreover, competitive equilibrium fails to be Pareto optimal; however, a suitably designed pay-as-you-go (PAYG) social security program can remove inefficiencies by allowing transfers from children to parents that are necessary for Pareto optimality but would not be possible in decentralized competitive equilibrium due to lack of individual incentives for such transfers (see for instance, Balasko and Shell [1981], Samuelson [1958]).

In another framework Becker [1974, 1981] establishes in his "Rotten Kid Theorem" that under certain circumstances when parents care about their children's welfare, children take actions that maximize the joint family income eventhough children do not care about their parents, provided parents leave positive bequest to their children. One implication of his Rotten Kid Theorem is that a forced transfer between children and parents have no ultimate effect on equilibrium outcome since parents can off-set this forced intergenerational transfer by suitably adjusting their bequest level.[2] Barro [1974] uses the above kind of intergenerational altruism in an OLG framework and shows that social security has no effect on savings so long as in equilibrium agents leave positive bequest in all periods. Furthermore, since Barro model is equivalent to one with finite number of infinitely lived agents, a competitive equilibrium is Pareto optimal; hence social security is not required for the purpose of attaining Pareto optimality of equilibrium allocation.

Both strands of above literature do not explain why transfers from children to parents are observed in many economies, and why the amount of transfers declines with the introduction of public transfer policies; why a PAYG social security program exists, and whether it is possible for the current living generations to legislate a PAYG social security benefits scheme for the current and all future generations such that the future generations will have no incentives to amend it; and if one such program exists does it lead to optimal allocation?

A few attempts have been made, however, to explain the existence of PAYG social security programs in frameworks that treat fertility exogenously. One type of explanations postulate that there could be economy of scale and other sources of market failures in pension provision (see, Diamond [1977]) or there might be adverse selection/moral hazard problems in private provision of retirement income insurance and these could be mitigated by compulsory participation (see Diamond and Mirrlees [1978]). These can explain introduction of fully funded system but cannot explain the existence of PAYG system.

Among the other type of explanations, Browning [1975] considers a voting model of social security in an OLG framework in which the old outvote the young

[2] See Bernheim, Shleifer and Summers [1985] for a critique of the Rotten Kid Theorem.

to enact a PAYG social security system. It is not, however, clear in Browning's framework why then the old do not use their power to enact a legislation to extract all income from the young. Hansson and Stuart [1989] provide an alternative explanation by modelling PAYG social security legislation as a trade among living generations. They consider an OLG model in which agents are assumed to derive utility not only from their own young age and old-age consumption but also from properly discounted young age and old-age consumption of their parents and of all future generations. They find conditions such that the young and old agents unanimously agree upon a stream of PAYG social security transfers for the current and all future generations such that the resulting allocation is Pareto optimal and that no future generations have incentives to amend the program.

Veall [1986] provides an alternative explanation for PAYG program by considering an OLG model in which each agent is assumed to derive utility not only from his/her own life-cycle consumption, but also from the level of old-age consumption of his/her parents. Due to this consumption externality, elderly may save little to extract the maximum possible gifts from their children; "This can lead to an inferior steady state, where no one is consuming 'enough' in retirement" (Veall [1986, p.250). If a PAYG social security system is introduced such that it transfers from the young to the old at least the amount that the old could extract from their children by saving nothing, such a social security program could restore inter-temporal efficiency of consumption for each agent and Pareto optimality for the whole society. However, once the agents begin to save, the young may like to reduce their social security contribution and have incentive to amend the PAYG social security legislation. Thus such a PAYG system may not be stable. Veall shows that if social security benefits are set at the level of optimal steady-state old-age consumption, then such a legislation will be honored by all future generations and thus is stable. Moreover, the resulting allocation will be Pareto optimal.

If agents expect to receive gifts from their children to support old-age consumption, it is clear that not only savings decisions but also the fertility decisions will be affected; in fact, agents would like to have more children.[3] Empirical analyses of cross country data as well as household survey data predominantly show that social security affects both fertility level and savings rate (see for instance, Nugent [1985] for a summary of these studies). Hence, it is important to relax the exogenous fertility assumption in the above class of models.

In more recent models that study effects of social security on fertility and savings (Barro and Becker [1989] and Raut [1992]) the existence of social security

[3] This is an alternative formulation of old-age security hypothesis.

is not explained. Nishimura and Zhang [1992] include fertility choices in Veall's one-sided altruism framework. Following Veall, they view the optimal old-age consumption in the steady-state as PAYG social security benefits. However, when fertility is also a choice variable, it is not possible to implement the optimal steady-state allocation using only a PAYG social security policy instrument; this was possible in Veall's framework because he treated fertility as exogenous; in fact, once such a PAYG social security program is enacted, the free rider's problem will cripple the system since an individual agent will have no incentive to have children (as they do not affect utility but cost money) and would like to depend on others' children to contribute to social security program. Since every body would do the same, such a social security program is not individually rational. Therefore, viewing optimal steady-state gifts as a form of PAYG social security in Veall's framework loses both normative and positive virtues once fertility is a choice variable.

In this paper, we extend Veall's framework to rectify some of these problems. We assume that agents derive utility not only from their own young age and old age consumption, but also from old-age consumption of their parents and the young age consumption of their children weighted by the number of children. This allows us to endogenize within family transfers in both directions (i.e., from children to parents as gifts and from parents to children as bequest); moreover, in our framework even when parents do not receive any old-age support they have individual incentives to have children.

In the overlapping generations framework decisions are made sequentially: in any given period, decisions regarding fertility, savings and intergenerational transfers of past generations and of the currently alive old generation that are made in the past are known to the current decision makers. When agents make their decisions they use all available information. Moreover, since agents know that their actions are observed by their children and hence will affect their children's decisions, they will take into account the incentive effects of their decisions on their children, and thus try to manipulate their children to get the best out of them. For instance, if an agent saves more for his retirement, then his children will transfer less income to the agent when he retires. Since the agent knows that his children react that way to his savings decisions, he might find it strategically advantageous to save little and have more children to extract higher transfers from his children. Thus, it is more natural to formulate our problem in a multi-stage game framework and apply the notion of subgame perfection to characterize the outcome of parents' manipulations. Much of the previous literature in this area ignores the sequential nature of the above overlapping decisions and apply the notion of open loop Nash equilibrium to characterize equilibrium outcomes. In open loop Nash equilibrium, agents take the actions of other agents as given but not their reactions and thus do not take proper account of the incentives that they face. Open loop Nash equilibrium makes sense only when agents must commit to entire time paths of decisions without observing anyone else's. In most models, open loop Nash equilibrium is easier to compute and hence it is often used as a benchmark to compare with other concepts of equilibrium.

In section 2, we set up our basic model and discuss the nature of coordination problems that the agents face, and compute the open loop Nash equilibrium as a benchmark for subgame perfect equilibrium. In section 3, we point out more formally the limitations of applying the open loop Nash equilibrium in our framework and reformulate the decision making of agents in a sequential multi-stage game in extensive form framework. We compute a subgame perfect equilibrium and study its properties and implications for social security. Section 4 concludes the paper.

2 Basic Framework

We use the basic framework in Raut [1991, 1992] and introduce two-sided altruism to endogenize intergenerational transfers. Let us assume that time is discrete and is denoted as $t = 0$, 1, 2, ...; each person lives for three periods: young, adult, and old. While young he is dependent on his parents for all decisions. We follow the convention that a superscript t refers to an adult of period t and a subscript t refers to time period t. For instance, c_t^t and c_{t+1}^t denote respectively the adult age and old-age consumption of an adult of period t; however, n_t denotes the number of children of an adult of period t, since we assume that only adults can have children, so from the subscript of n_t we can identify which generation it corresponds to.

We assume that for all $t \geq 0$, the wage rate w_t and the interest rate r_{t+1} which are faced by the adults of generation t, are exogenously given.

2.1 Households

We assume that all children are born identical and they all behave identically in a given situation. We would like to derive agent's behavior regarding fertility, savings and intergenerational transfers from utility maximization. We model an individual's concern for his parents and children by assuming that an adult of generation t derives utility from his own life-cycle consumption and from consumption level of his children and parents that he observes during his active life-time (for a justification of these type of utility functions, see Pollak [1988]). More specifically we postulate the following utility function:

$$W_t \ = \ \delta(n_{t-1})\mathrm{v}(c_t^{t-1}) + \alpha\mathrm{v}(c_t^t) + \beta\mathrm{v}(\ c_{t+1}^t) \ + \ \gamma(n_t)\mathrm{v}(c_{t+1}^{t+1}) \tag{1}$$

Veall [1986] in his exogenous fertility framework and Nishimura and Zhang [1992] in their endogenous fertility framework assumed that $\gamma(n_t) = 0$ and $\delta(n_t) =$constant, for all $t \geq 0$. When there are many siblings, an individual may not care about his parents as intensely as he would do if he were the only child. In the above specification of utility function, we allow the degree of an individual's concern for his parents to depend on the number of siblings. However, much of our results hold if $\delta(.)$ is constant.

In our economy, agents have interdependent utility functions: an agent's utility is affected by the amount of consumption of other family members. Thus, the agents have incentives to transfer part of their income to their parents and children. There are several difficulties in modelling the coordination of these interdependent transfer decisions. The coordination problem that a representative adult of period t, $t \geq 1$ faces is as follows:

An adult of period t earns wage income w_t in the labor market and *expects* to receive a bequest b_t from his parents. These two sources of income constitute his budget during adulthood. Rearing cost per child in period t is $\theta_t > 0$ units of period t good. Given his adulthood budget, he decides the amount of savings s_t, the number of children $n_t \geq 0$, the fraction of income to be transferred to his old parents $a_t \geq 0$; in the next period, he retires and expects to receive $a_{t+1}n_t$ amount of gifts from his children, earns $(1 + r_{t+1})s_t$ as return from his physical assets, and decides the amount of bequest $b_{t+1} \geq 0$ to leave for each of his children. Moreover, agent t's t-th period decisions, (a_t, n_t, s_t) overlap with his parent's bequest decision, b_t; similarly, his bequest decision, b_{t+1}, overlaps with his children's gift decisions, a_{t+1}. The time structure of overlapping decisions is shown in table 1.

The effects of agent t's action, $\alpha^t = (a_t, n_t, s_t, b_{t+1})$, on the levels of his own life cycle consumption and the levels of consumption of his parents and children in the periods that overlap with his life-cycle, depend on his parent's action, α^{t-1} and his children's action α^{t+1} as follows:

$$c_t^t + s_t + \theta_t n_t \quad = \quad (1 - a_t)w_t + b_t \tag{2}$$

$$c_{t+1}^t + n_t b_{t+1} \quad = \quad (1 + r_{t+1})s_t + a_{t+1}w_{t+1}n_t \tag{3}$$

$$c_t^{t-1} \quad = \quad (1 + r_t)s_{t-1} - n_{t-1}b_t + a_t w_t n_{t-1} \tag{4}$$

Table 1: Time table of actions by overlapping generations of agents

time generation	$t=0$	$t=1$	$t=2$	$t=3$	...	$t-1$	t	...
-1	b_0							
0	(a_0, n_0, s_0)	b_1						
1		(a_1, n_1, s_1)	b_2					
2			(a_2, n_2, s_2)	b_3	...			
...					...			
$t-1$						$(a_{t-1}, n_{t-1}, s_{t-1})$	b_t	
t							(a_t, n_t, s_t)	...
...					...			...

$$\begin{aligned} c_{t+1}^{t+1} &= (1-a_{t+1})w_{t+1} + b_{t+1} - s_{t+1} - \theta_{t+1} n_{t+1} \qquad (5) \\ & c_t^t, c_{t+1}^t \geq 0 \end{aligned}$$

Let agent t's choice vectors be in the set, $S^t \subset \Re_+^4$, defined by

$$S^t = \left\{ \begin{array}{lcl} \alpha^t = (a_t, n_t, s_t, b_{t+1}) \in \Re_+^4 & | & c_t^t, c_{t+1}^t \text{ defined in equations } (2),(3) \\ & & \text{are} \geq 0 \text{ with } b_t = 0 \text{ and } a_{t+1} = 0 \end{array} \right\}$$

Similarly, the agent $t = 0$'s utility function is given by

$$W_0 = \beta \mathrm{v}(c_1^0) + \gamma(n_0)\mathrm{v}(c_1^1)$$

and agent $t = 0$ decides the level of bequest b_1, given his past decisions, n_0, s_0, and his children's decisions, α^1. The arguments of his utility function are given by

$$\begin{aligned} c_1^0 + n_0 b_1 &= (1+r_1)s_0 + a_1 w_1 n_0 \qquad (6) \\ c_1^1 &= (1-a_1)w_1 + b_1 - s_1 - \theta_1 n_1 \qquad (7) \\ & c_1^0 \geq 0 \end{aligned}$$

His set of choice vectors, $S^0 \subset \Re_+$ can be defined as

$$S^0 = \left\{ b_1 \geq 0 \mid (6) \text{ is satisfied with } a_1 = 0, c_1^0 \geq 0 \right\}$$

Almost all previous studies applied a version of an *open-loop Nash equilibrium* concept, which is defined as the set of strategies, $\{\alpha^t \mid \alpha^t \in S^t, t \geq 0\}$ such that for no $t \geq 0$, there exists a $\tilde{\alpha}^t \in S^t$ such that given $\alpha^\tau, \tau \neq t$, the consumption vector from $\tilde{\alpha}_t$ yields higher utility for agent t than from α^t (see Fudenberg and Tirole [1991] for the concept of open-loop Nash equilibrium).

Note that if $1 > a_t^* > 0$ and $b_t^* > 0$ is an open loop Nash equilibrium combination of gifts and bequest in period t, so is $a_t^* + \epsilon$ and $b_t^* + \epsilon w_t$, for small $\epsilon > 0$; this can lead to gift-bequest war or the tragedy of miscoordination as in the gift of magi.[4] This could be handled by restricting to open loop Nash equilibria that yield either positive bequest or positive gift within a period but not both.

Another problem with the open-loop Nash-equilibrium is that given open loop Nash equilibrium levels of gifts from his children and bequest from his parents[5], while there may not exist a feasible strategy $\tilde{\alpha}_t$ in S^t yielding higher utility for any agent t, $t \geq 0$, there may exist $\hat{\alpha}^t$ outside S^t that satisfies the budget constraints (2)-(5) [or (6)-(7) for $t = 0$] yielding higher utility for agent t, for some $t \geq 0$.

These are not very serious problems and could be avoided by restricting the open-loop Nash equilibrium as follows:

For given $b_{t-1}, s_{t-1}, n_{t-1},\ a_{t+1}$, s_{t+1}, and n_{t+1}, a vector (n_t, s_t, b_{t+1}, a_t, c_t^t, c_{t+1}^t, c_t^{t-1}, $c_{t+1}^{t+1}) \geq 0$ is *feasible from the perspective of agent* $t \geq 1$ if it satisfies the budget constraints (2)-(5) of the above maximization problem. For given s_0, n_0, a_1, s_1, and n_1, a vector $(b_1, c_1^0, c_1^1) \geq 0$ is *feasible from the perspective of agent* $t = 0$ if it satisfies the constraints (6)-(7).

Definition 2.1 A*n open loop Nash equilibrium* is a sequence $\{(a_t, b_t, s_t,\ n_t,\ c_t^t,$ $c_{t+1}^t)\}_{t=1}^\infty$, c_1^0 such that for given initial condition, n_0, s_0,

(i) $a_t > 0 \Rightarrow b_t = 0$ and $b_t > 0 \Rightarrow a_t = 0$

(ii) for $t \geq 1$, given $\alpha^{t-1} = (a_{t-1}, n_{t-1}, s_{t-1}, b_t)$ and $\alpha^{t+1} = (a_{t+1}, n_{t+1}, s_{t+1}, b_{t+2})$ there does not exists a feasible choice vector $(\tilde{n}_t$, $\tilde{s}_t$, $\tilde{b}_{t+1}$, $\tilde{a}_t$, $\tilde{c}_t^t$, $\tilde{c}_{t+1}^t$, $\tilde{c}_t^{t-1}$, $\tilde{c}_{t+1}^{t+1})$ from agent t's perspective that yields higher utility for him. Similarly, for $t = 0$, and given n_0, s_0, and given $\alpha^1 = (a_1, n_1, s_1) \in S^1$ there does not exist another feasible choice vector, $(\tilde{b}_1, \tilde{c}_1^0, \tilde{c}_1^1)$ from the perspective of agent $t = 0$ that yields higher utility for him.

[4] Although in O'Henry's story both parties were made worse-off because of the gift exchange, in our model, while there is mis-coordination of the gift and bequest decisions of the agents within a period, there is, however, no welfare loss due to such miscoordination of decisions.

[5] The latter does not apply if $t = 0$

There is, however, a serious deficiency in the open loop Nash equilibrium as an adequate characterization of the incentives that the agents face in our set-up.

An open loop Nash equilibrium assumes that each agent takes the actions of other agents as given. At an equilibrium, there might be scope for agents to manipulate their parents' or their children's behavior to extract more transfers from them. For instance, since parents make their consumption and fertility decisions prior to their children's, parents may find it strategically advantageous to consume more in their working age, save little on physical assets and possibly have more children so that when they become old they have little income of their own. When the children find that their old parents have little to consume, they will have sympathy for their parents since they care about their parents' consumption; thus they will transfer a larger amount of old-age support than what they would be transferring in the open loop Nash equilibrium. The children in turn can manipulate their children in the same way and be better-off as a result. This process might be self-fulfilling over time.

We model such manipulations in a later section by reformulating the above coordination problem as a multi-stage game in extensive form and use subgame perfect Nash equilibrium as the equilibrium outcome of manipulation. In the rest of this section, we compute open loop Nash equilibria as benchmarks with which the subgame perfect Nash equilibria of the multi-stage game are compared.[6]

2.2 Characterization of Open Loop Equilibria

Assume that an open loop Nash equilibrium exists and that the instantaneous utility function, v(c), satisfies Inada condition so that unrestricted maximization of W_t with respect to c_t^t and c_{t+1}^t always yields positive consumption. The equilibrium will satisfy the following first order necessary conditions of the parents' optimization problems:

Corresponding to agent $t = 0$'s optimization problem we have

$$-\beta n_0 \mathrm{v}'(c_1^0) + \gamma(n_0)\mathrm{v}'(c_1^1) \leq 0 \text{ and } = 0 \text{ if } b_1 > 0 \tag{8}$$

corresponding to any other agent t's ($t \geq 1$)optimization problem:

$$\delta(n_{t-1})\mathrm{v}'(c_t^{t-1})w_t n_{t-1} - \alpha \mathrm{v}'(c_t^t)w_t \leq 0 \text{ and } = 0 \text{ if } a_t > 0 \tag{9}$$

[6] Several other equilibrium concepts have been proposed in the literature in this situation, see Raut [1990a] for the concept of Lindahl equilibrium, and Pollak [1988] for other concepts. However all these concepts are in models with exogenous fertility.

$$-\alpha v'(c_t^t) + \beta v'(c_{t+1}^t)(1 + r_{t+1}) \leq 0 \text{ and } = 0 \text{ if } s_t > 0 \tag{10}$$

$$-\alpha v'(c_t^t)\theta_t + \beta v'(c_{t+1}^t)(a_{t+1}w_{t+1} - b_{t+1}) + \gamma'(n_t)v(c_{t+1}^{t+1}) \leq 0, \text{ and } = 0 \text{ if } n_t > 0 \tag{11}$$

$$-\beta v'(c_{t+1}^t)n_t + \gamma(n_t)v'(c_{t+1}^{t+1}) \leq 0 \text{ and } = 0 \text{ if } b_{t+1} > 0 \tag{12}$$

At $t = 1$, either $a_1 > 0$, in which case (9) is an equality (at $t = 1$) from which we calculate a_1 and then check if the inequality (8) is satisfied; or $b_1 > 0$, in which case (8) is an equality from which we calculate b_1 and then check if the inequality (9) is satisfied. There are situations when neither of the above is true, and hence there may not exist an open loop equilibrium. For instance, suppose parents care too much about their children's adult-age consumption as compared to their own old-age consumption and children care too much about their parent's old-age consumption as compared to their own adult-age consumption. Or in other words, suppose α and β in (1) are close to zero, then parents would like to transfer their income to their children but children would not accept it, on the other hand, children would like to give a gift to their parents but parents would not accept it.

We further distinguish among different types of equilibria. *An open-loop bequest equilibrium* is an equilibrium of the above type that satisfies $a_t = 0$, and $b_t > 0$ for all $t \geq 1$. *An open-loop gift equilibrium* is an equilibrium of the above type that further satisfies $b_t = 0$, and $a_t > 0$ for all $t \geq 1$. Similary, an *open-loop equilibrium with no transfers* is one in which $b_t = a_t = 0$ for all t ≥ 0. There could be also equilibria in which bequests are operative in some periods and gifts are operative in other periods. In this paper we will analyze only open-loop gift equilibria. It could be shown from the above first order conditions that in general there is indeterminacy in the set of open loop equilibria. This indeterminacy is symptomatic of Nash equilibria with interdependent utility functions. For our purpose, we focus on steady-state open loop gift equilibria which are determinate.

2.3 Steady-state Open Loop Gift Equilibria

A *steady-state open loop gift equilibrium* is an open loop gift equilibrium such that $a_t = a^* > 0$, $n_t = n^* > 0$, $s_t = s^* \geq 0$ and $b_t = 0$ for all t and (8)- (12) are satisfied.

We denote all steady-state endogenous variables with a $*$, and drop the time scripts. We assume that $w_t = w^*$, $r_t = r^*$ and $\theta_t = \theta$ for all $t \geq 1$. Since this stationarity assumption is not critical to the issues of the paper, to simplify exposition, we will maintain this assumption in the rest of the paper. Let us denote by c_1^* and c_2^* respectively the adult age and old-age consumption in the steady-state.

Thus, for a steady-state gift equilibrium, we have $c_1^* \equiv (1-a^*)w^* - \theta n^* - s^*$ and $c_2^* \equiv (1+r^*)s^* + w^* a^* n^*$. The first order necessary conditions, (9)-(12), for such an equilibrium simplify to

$$\frac{\mathrm{v}'(c_2^*)}{\mathrm{v}'(c_1^*)} = \frac{\alpha}{\delta(n^*)n^*} \tag{13}$$

$$\frac{\mathrm{v}(c_1^*)}{\mathrm{v}'(c_1^*)} = \frac{\alpha}{\gamma'(n^*)}\left[\theta - \frac{\beta a^* w^*}{\delta(n^*)n^*}\right] \tag{14}$$

$$\frac{\mathrm{v}'(c_2^*)}{\mathrm{v}'(c_1^*)} \geq \frac{\gamma(n^*)}{\beta n^*} \tag{15}$$

$$1 + r^* \leq \frac{\delta(n^*)n^*}{\beta}, (\text{equality if s}^* > 0) \tag{16}$$

The following proposition summarizes the properties of open loop gift equilibria.

Proposition 1 *Let $\delta(n)n$ be increasing in n. Let (n_0^*, a_0^*, U_0^*) be the vector of fertility, gifts and utility levels corresponding to a steady-state gift equilibrium with $s^* = 0$ and (n_s^*, a_s^*, U_s^*) be the corresponding vector for a steady-state gift equilibrium with $s^* > 0$, then $n_0^* \geq n_s^*$, and $U_0^* \geq U_s^*$, the latter being a strict inequality when the no-bequest constraint (15) is a strict inequality.*

PROOF: From (16) we have for a steady-state gift equilibrium with $s^* = 0$, $\delta(n_0^*)n_0^*/\beta \geq 1 + r^*$ and for a steady-state gift equilibrium with $s^* > 0$, we have $\delta(n_s^*)n_s^*/\beta = 1 + r^*$. Combining these two, we have $\delta(n_0^*)n_0^* \geq \delta(n_s^*)n_s^*$, i.e., $n_0^* \geq n_s^*$.

The proof of the second part follows from proposition 4 in the next section.

Q.E.D.

In the following example we show the coexistence of unique steady-state open loop gift equilibria of two types: one type with $s^* = 0$ and the other type with $s^* > 0$.

2.4 An Example

The instantaneous utility function satisfies the following:

Assumption A: 2.1 (constant elasticity of marginal utility (CEM) function)

$$v(c) = \frac{c^{1-\rho}}{1-\rho},\ \rho \neq 1,\ 0 < \rho < \infty \tag{17}$$

where $-\rho$ measures the elasticity of marginal utility.

Assumption A: 2.2 $\gamma(n) = \gamma_0 n^{1-\gamma_1},\ 0 \leq \gamma_1 < 1$

The significance of this assumption is that parents care about consumption of all children equally. However, the weights they give to such consumption decrease with the number of children whenever $\gamma_1 > 0$.

Assumption A: 2.3 $\delta(n) = \delta_0 n^{\delta_1 - 1},\ 0 \leq \delta_1 \leq 1$

Two types of steady-state gift equilibria may coexist. Let us first find steady-state gift equilibria with $s^* > 0$. Equation (16) determines the steady-state equilibrium n_s^* uniquely and equations (13) and (14) reduce to the following two linear equations:

$$s = \frac{\mu(w^* - \theta n_s^*) - w^*(n_s^* + \mu)a}{1 + r^* + \mu} \tag{18}$$

$$s = \frac{(1-\rho)\alpha\theta\mu n_s^{*\gamma_1}}{(1+r^*)\gamma_0(1-\gamma_1)} - \frac{(1+r^*)\gamma_0(1-\gamma_1)w^* n_s^{*1-\gamma_1} + (1-\rho)\alpha w^*\mu}{(1+r^*)^2\gamma_0(1-\gamma_1)}.n_s^{*\gamma_1} a \tag{19}$$

where $\mu = (\beta(1+r^*)/\alpha)^{1/\rho}$.

The linear equations (18) and (19) for the above set of parameter values are shown respectively as $s_1(a)$ and $s_2(a)$ in figure 1.

Notice that the intercept of equation (18) is always positive since the child cost, θn_s^* is less than wage income in gift equilibrium. The intercept of equation (19) is positive if $\rho < 1$, in which case the slopes are negative for both lines and we cannot guarantee that they will intersect in the positive orthant. However, if $\rho > 1$, equation (19) will have negative intercept and positive slope. If ρ is sufficiently larger than one, then it will intersect with the line (18), and we have unique steady-state gift equilibrium: $\delta_0 = .35$;$\delta_1 = .8$; $\gamma_0 = .3$, $\gamma_1 = .6$; $\rho = 1.5$; $\alpha = .4$; $\beta = .34$; $r^* = .05$; $w^* = 10$; and $\theta = .1$. The equilibrium quantities are as follows: $(n_s^*, s^*, a^*) = (1.025, 1.341, .334)$ and $(c_1^*, c_2^*, U_{max}) = (5.214, 4.833, -1.24)$; one can easily verify that (15) is satisfied with strict inequality.

Let us now examine how many steady-state gift equilibria exist when $s^* = 0$, and whether for the above set of parameters, such an equilibrium could be found.

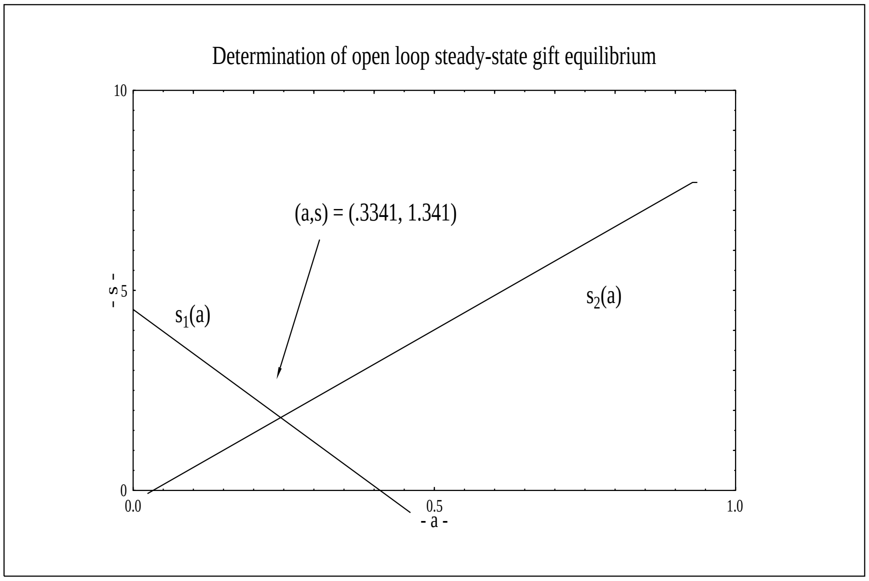

Figure 1: Determination of steady-state gift equilibrium

It can be shown easily that (13) and (14) simplify to the following two equations in two unknowns, a and n:

$$(1-a)w^* - \theta n = \left[\frac{\delta(n)n}{\alpha}\right]^{1/\rho} a w^* n \tag{20}$$

$$(1-a)w^* - \theta n = \frac{\alpha(1-\rho)}{\gamma'(n)}\left[\theta - \frac{\beta w^* a}{\delta(n)n}\right] \tag{21}$$

Equilibrium a_0^* and n_0^* is a solution of (20) and (21) that also satisfies (15) and (16). From the above implicit equations, it is not difficult to get a explicitly as a function of n, and let these functions be denoted as $a_1(n)$ and $a_2(n)$ respectively. The graphs of these two functions are shown in figure 2; it is clear that there exists only one solution $(n^*, s^*, a^*) = (1.6997$,0,.4096), $(c_1^*, c_2^*, U^*) = (5.734, 6.961, -1.1402)$; moreover (15) and (16) are satisfied as strict inequalities.

Comparing these two open-loop gift equilibria we find that the equilibrium with zero savings has higher levels of fertility, transfers from children and utility level than the gift equilibrium with positive savings. This shows that agents have incentive to manipulate their children. In the next section we model manipulation of children formally and compare its equilibrium outcome with the open loop gift equilibrium outcomes.

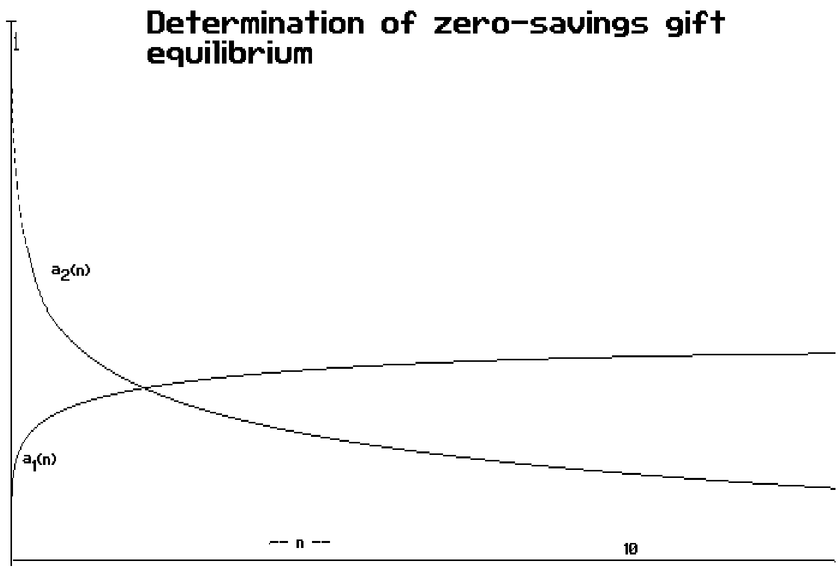

Figure 2: Determination of steady-state gift equilibrium

3 Manipulation and Subgame Perfection

From the time table of actions of various generations it is clear that a representative adult of period t has already made his decisions $(a_{t-1}, n_{t-1}, s_{t-1})$ which together with the decisions of all past generations are observable to himself and to his children. In period t, a representative agent t-1 decides b_t, and a representative agent t decides (a_t, n_t, s_t) both simultaneously; optimal decisions of the agents in period t depend on the information they already have; since agent t-1 knows that his children will use the information regarding his observable actions of the previous period, he will choose his action that exploits the reactions of his children in the most favorable way. Or in other words, parents may find it beneficial to manipulate their children's behavior. To analyze these issues, it is natural to use the framework of multi stage game with observed actions and the notion of subgame perfect equilibrium as described below.

We associate stage t with time period t. We are currently at time $t = 1$ when we are analyzing the economy. Let h_t denote the common information or history of all the actions that have been taken by all agents up to time t. h_t is defined recursively as follows:

$$\begin{aligned} h_1 &= (b_0, a_0, n_0, s_0) \text{ (initial condition)} \\ h_2 &= (h_1 | (b_1, a_1, n_1, s_1)) \\ &\ldots\ldots \\ h_t &= (h_{t-1} | (b_{t-1}, a_{t-1}, n_{t-1}, s_{t-1})), \forall t \geq 2 \end{aligned}$$

Let us denote by $\mathcal{H}_t$ the set of all possible histories up to time t. Let $S_t^{t-1}(h_t) \subset \Re_+$ be the set of feasible bequest decisions of agent t-1 (denoted as superscript t-1) at stage t (denoted as subscript t) defined by

$$S_t^{t-1}(h_t) = \left\{ b_t \geq 0 \mid (4) \text{ is satisfied with } c_t^{t-1} \geq 0,\ s_{t-1}, n_{t-1} \text{consistent with } h_t \right\}$$

Note that the above set of feasible bequest decisions depend on the history h_t, especially on the agent's own savings and fertility decisions. At stage t, agent t-1's actions are functions of the form $b_t : \mathcal{H}_t \rightarrow \Re$, such that $b_t(h_t) \in S_t^{t-1}(h_t)$.

Similarly, given the history h_t, $S_t^t(h_t) \subset \Re_+^3$, the set of feasible actions of an adult agent in stage t, is defined by

$$S_t^t(h_t) = \left\{ (a_t, n_t, s_t) \in \Re_+^3 \mid (2) \text{ is satisfied with } b_t = 0, c_t^t \geq 0 \right\}$$

At stage t, agent t's actions are functions, $(a_t, n_t, s_t) : \mathcal{H}_t \rightarrow \Re^3$ such that $(a_t, n_t, s_t)(h_t) \in S_t^t(h_t)$. Once agents t and t-1 choose their actions in period t, the history gets updated to h_{t+1}, and the game moves to stage $t+1$ in which agents t and $t+1$ are active and their feasible actions are defined exactly in the same fashion as in the previous stage. Let us denote the game starting at stage t with history h_t as $\Gamma(h_t)$. Figure 3 depicts a part of the extensive form of the game $\Gamma(h_t)$: the tree is shown only up to stage $t+2$; the label of a branch describes the action of the agent that it corresponds to; the shaded boxes are the information sets of the agents within a given stage. In this notation, the economy we are analyzing is represented by the game $\Gamma(h_1)$.

A Pure strategy of agent t is a vector

$$\sigma_t = \begin{cases} ((a_t, n_t, s_t)(h_t), b_{t+1}(h_{t+1})) \in S_t^t(h_t) \times S_{t+1}^t(h_{t+1}) \text{ such that} & \\ \qquad h_t \in \mathcal{H}_t, h_{t+1} = (h_t | (b_t, a_t, n_t, s_t)(h_t)) & if\ t \geq 1 \\ b_1(b_0, a_0, n_0, s_0) & if\ t = 0 \end{cases} \tag{22}$$

A *strategy profile of the game* $\Gamma(h_1)$ is a set of pure strategies of all the players, $\sigma = \{\sigma_t\}_{t=0}^{\infty}$. For any history h_t up to stage t and for any $\tau \geq t$, define $\mathcal{H}_\tau(h_t)$ as

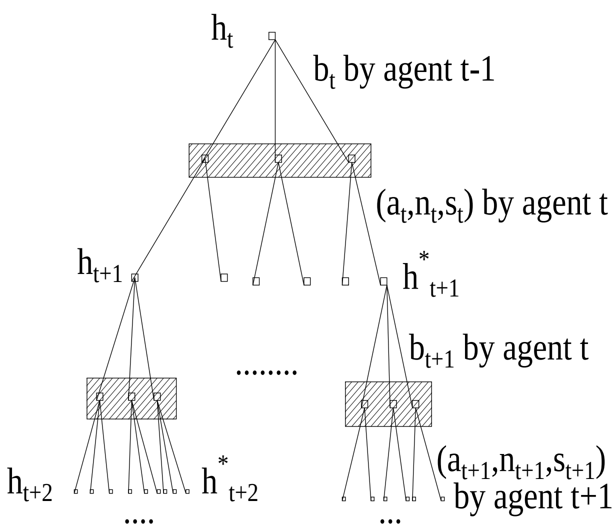

Figure 3: Extensive form representation of the multi-stage game, $\Gamma(h_t)$

the set of all possible histories up to stage τ starting from the common history h_t at stage t. Given a history h_t up to stage t ($t \geq 0$), and corresponding to a strategy profile $\sigma = \{\sigma_t\}_0^\infty$ as in (22), we define a strategy profile $\sigma(h_t) = \{\sigma_\tau(h_t)\}_{\tau=t-1}^\infty$ for the subgame $\Gamma(h_t)$ by

$$\sigma_\tau(h_t) = \begin{cases} ((a_\tau, n_\tau, s_\tau)(h_\tau), b_{\tau+1}(h_{\tau+1})) \in S_\tau^\tau(h_\tau) \times S_{\tau+1}^\tau(h_{\tau+1}) \text{ such} & \\ \quad \text{that } h_\tau \in \mathcal{H}_\tau(h_t) \text{ and } h_{\tau+1} = (h_\tau|(b_\tau, a_\tau, n_\tau, s_\tau)(h_\tau)) & if\ \tau \geq t \\ b_t(h_t) & for\ player\ t-1 \end{cases} \tag{23}$$

In the above notation, $\sigma_\tau(h_0) \equiv \sigma_\tau$, for all $\tau \geq 0$. Note that $\{\sigma_\tau(h_t)\}_{\tau=t}^\infty$ is a well defined profile of strategies for the game $\Gamma(h_t)$.

Definition 3.1 *A* subgame perfect equilibrium *starting at an initial condition* b_0, a_0, n_0 *and* s_0 *is a profile of strategies* $\{\sigma_t\}_0^\infty$ *defined in (22) such that* $\{\sigma_\tau(h_t)\}_{\tau=t}^\infty$ *defined in (23) is a Nash equilibrium of the game* $\Gamma(h_t)$ *for all* $h_t \in \mathcal{H}_t$, $t \geq 0$.

In the above set-up, agents in later stages can use very complex punishment rules as their strategies. For instance, an agent $t = 5$ in stage 5 can condition his actions as follows: ”he will transfer a certain fraction a_5 of his income to his his parents if his parents transferred a certain fraction a_4 of their income to the agent's grandparents, saved certain amount s_4, had certain number of children, n_4, and if his grand parents transferred a certain fraction a_3 of their income to the agent's grand grand parents, ... so on.” While these types of strategies may lead to many subgame perfect equilibria, the equilibria that prescribe strategies conditioning on the dead grand parents are hard to execute since it is not possible to objectively verify if the agent's grand parents or grand grand parents did such and such things.

Using the Markovian structure of our economy, and the fact that utility functions depend only on parent's old-age and the children's young age consumption, we can take as focal point a subgame perfect equilibrium that conditions only on the actions that are observable within an agent's life time. More specifically, note that $S_t^t(h_t)$ does not depend upon history h_t and $S_t^{t-1}(h_t)$ depends only on agent t-1's own past decisions. From equations (2)-(5), and the arguments of the utility function, it is clear that the only information from history that is relevant to decision making of the agents in stage t are agent t-1's own past decisions $(a_{t-1}, s_{t-1}, n_{t-1})$ in making his bequest decision b_t, and (s_{t-1}, n_{t-1}) in making agent t's decisions, (a_t, n_t, s_t). Thus the agent t's strategies are functions of the type: $a_t = a_t(n_{t-1}, s_{t-1})$, $n_t = n_t(n_{t-1}, s_{t-1})$, and $s_t = s_t(n_{t-1}, s_{t-1})$ which are known as *reaction functions*. Utilizing the envelop theorem, we note that when agent t jointly determines $a_t(.,.)$, $n_t(.,.)$, $s_t(.,.)$, of stage t and $b_{t+1}(.,.,.)$ of stage $t+1$, he can treat b_{t+1} as scalar. Putting all the actions and reactions of agent t from all stages of the game together, his strategy is now given by an infinite dimensional vector in function space as follows:

$$\mathcal{A}_t = \begin{cases} (a_t(.,.), n_t(.,.), s_t(.,.), b_{t+1}) & \text{if } t \geq 1 \\ b_1(a_0, n_0, s_0) & \text{if } t = 0 \end{cases}$$

Note that a_t, s_t, and n_t now belong to function spaces, whereas in open loop Nash equilibrium they were non-negative real numbers. Also note that in our context the subgame starting at $\Gamma(h_t)$ depends only on the components, $(b_{t-1}, a_{t-1}, n_{t-1}, s_{t-1})$ of the history; we will denote this subgame as $G(b_{t-1}, a_{t-1}, n_{t-1}, s_{t-1})$ instead of $\Gamma(h_t)$. The following proposition can be proved easily.

Proposition 2 *Let the initial condition be given by b_0, a_0, n_0 and s_0. Let the sequence of strategies $\mathcal{A}^*_t = (a^*_t(n_{t-1}, s_{t-1}), s^*_t(n_{t-1}, s_{t-1}), n^*_t(n_{t-1}, s_{t-1}), b^*_{t+1})$, $t \geq 1$, and $\mathcal{A}^*_0 = (a_0, n_0, s_0, b^*_1)$ be such that $\{\mathcal{A}^*_{t+\tau}\}^{\infty}_{\tau=0}$ is a Nash equilibrium of the game $G(b_{t-1}, a_{t-1}, n_{t-1}, s_{t-1})$, for all t ≥ 1. Then $\{\mathcal{A}^*_t\}^{\infty}_0$ is a subgame perfect equilibrium.*

The difference between a subgame perfect Nash equilibrium and an open loop Nash equilibrium is that in the latter, agent t takes his children's gifts a_{t+1} and parent's bequest decision b_t as given, whereas in a subgame perfect Nash equilibrium, he takes his parent's bequest decision, b_t and the reaction functions of his children, $a_{t+1}(n_t, s_t)$, $n_{t+1}(n_t, s_t)$ and $s_{t+1}(n_t, s_t)$ as given when he decides on the number of children, n_t, and the amount of savings, s_t.

Similar to the case of open loop Nash equilibrium, we can define subgame perfect gift equilibrium and subgame perfect bequest equilibrium. However, in the rest of the paper we analyze only the properties of the subgame perfect gift equilibria.

3.1 Conditions characterizing subgame perfect gift equilibria

Let $a_{t+1}(.,.)$, n_{t+1}(.,.), $s_{t+1}(.,.)$ be the optimal reaction functions of agent $t+1$, and let n_{t-1}, s_{t-1} be any feasible actions of agent $t-1$. Taking these decisions as given, agent t chooses a feasible $\mathcal{A}_t = (a_t(.,.)$, s_t(.,.), $n_t(.,)$, $b_{t+1})$ that maximizes his utility. For $t \geq 1$, the first order necessary conditions for his maximization problem are as follows:

$$-\alpha \mathrm{v}'(c_t^t) + \beta \mathrm{v}'(c_{t+1}^t)\left[(1+r_{t+1}) + w_{t+1} n_t a_{t+1,2}(n_t, s_t)\right] - \gamma(n_t)\mathrm{v}'(c_{t+1}^{t+1}) \times$$
$$[a_{t+1,2}(n_t, s_t) w_{t+1} + s_{t+1,2}(n_t, s_t) + \theta_{t+1} n_{t+1,2}(n_t, s_t)] \leq 0 \text{ and } = 0 \text{ if } s_t > 0 \quad (24)$$

$$-\alpha \theta_t \mathrm{v}'(c_t^t) + \beta \mathrm{v}'(c_{t+1}^t)\left[a_{t+1}(n_t, s_t) w_{t+1} + n_t w_{t+1} a_{t+1,1}(n_t, s_t)\right] + \gamma'(n_t)\mathrm{v}(c_{t+1}^{t+1})$$
$$-\gamma(n_t)\mathrm{v}'(c_{t+1}^{t+1})\left[w_{t+1} a_{t+1,1}(n_t, s_t) + s_{t+1,1}(n_t, s_t) + \theta_{t+1} n_{t+1,1}(n_t, s_t)\right] = 0 \quad (25)$$

$$\delta(n_{t-1})\ \ n_{t-1}\mathrm{v}'([1+r_t]s_{t-1} + a_t w_t n_{t-1}]) - \alpha \mathrm{v}'([(1-a_t)w_t - s_t - \theta_t n_t]) = 0 \quad (26)$$

$$-\beta \mathrm{v}'(c_{t+1}^t) n_t + \gamma(n_t)\mathrm{v}'(c_{t+1}^{t+1}) \leq 0 \text{ and} = 0 \text{ if } b_{t+1} > 0 \quad (27)$$

Could we use the above conditions to determine subgame perfect equilibrium choices of the agents? If we knew the explicit form of $a_{t+1}(.,.)$, $n_{t+1}(.,.)$ and $s_{t+1}(.,.)$ then equations (24) - (26) provide a system of implicit functions $\Phi(n_{t-1}, s_{t-1}, a_t, n_t, s_t) = 0$, where Φ is a three dimensional vector of functions. Assuming suitable conditions (such as v is twice continuously differentiable etc.), we could get differentiable solutions $a_t(n_{t-1}, s_{t-1})$, $n_t(n_{t-1}, s_{t-1})$, $s_t(n_{t-1}, s_{t-1})$ and $b_{t+1} = 0$. If furthermore, $a_{t+1}(.)$, $n_{t+1}(.)$ and $s_{t+1}(.)$ are chosen such that for all n_{t-1} and s_{t-1} the solution of $\Phi(.) = 0$ is a global maximum of agent t's utility maximization problem, then yes we could use these equations to determine subgame perfect equilibrium. However, we do not know the form of the functions as assumed and thus we cannot use the system of equations (24)-(26) to find the optimal reaction functions iteratively in the above way. We can, however, use the above system of equations to find steady-state subgame perfect gift equilibria as follows:

A *steady-state subgame perfect gift equilibrium* is a vector $(n^*, s^*, a(.,.), n(.,.), s(.,.))$ such that

$$\begin{aligned}
a_t(n_{t-1}, s_{t-1}) &= a(n_{t-1}, s_{t-1}) \\
n_t(n_{t-1}, s_{t-1}) &= n(n_{t-1}, s_{t-1}) \\
s_t(n_{t-1}, s_{t-1}) &= s(n_{t-1}, s_{t-1}) \\
b_t &= 0 \ \text{ for all } t \geq 1
\end{aligned}$$

and

$$n^* = n(n^*, s^*),\ s^* = s(n^*, s^*)$$

and that the above satisfies the system of equations (24)-(26) for all $t \geq 1$ with initial condition, $n_0 = n^*$, and $s_0 = s^*$.

There may exist many steady-state subgame perfect gift equilibria. To use the above considions to find some equilibria let us assume that $n(n_{t-1}, s_{t-1}) = n_{t-1}$ and $s(n_{t-1}, s_{t-1}) = s_{t-1}$. Note that for such reaction functions, we have $n_1 = 1$, $n_2 = 0$, $s_1 = 0$ and $s_2 = 1$. Let c_1^* and c_2^* be the steady-state subgame perfect equilibrium consumption during adult age and old-age of an agent. The system of equations (24)-(26) for a steady-state subgame perfect equilibrium becomes:

$$-\alpha \mathrm{v}'(c_1^*) + \beta \mathrm{v}'(c_2^*)\left[(1+r) + wna_2(n,s)\right] - \gamma(n)\mathrm{v}'(c_1^*) \times \\ \left[a_2(n,s)w + 1\right] \leq 0 \text{and} = 0 \text{ if s} > 0 \tag{28}$$

$$-\alpha\theta \mathrm{v}'(c_1^*) + \beta \mathrm{v}'(c_2^*)\left[a(n,s)w + a_1(n,s)wn\right] + \gamma'(n)\mathrm{v}(c_1^*) \\ -\gamma(n)\mathrm{v}'(c_1^*)\left[a_1(n,s)w + \theta\right] = 0 \tag{29}$$

$$\frac{\delta(n)n}{\alpha} = \frac{\mathrm{v}'\left([1-a(.)]w - s - \theta n)\right)}{\mathrm{v}'\left((1+r)s + a(.)wn\right)} \tag{30}$$

The above system is augmented by the no bequest condition (15). We then solve for $a(.,.)$ from equation (30) and then s^* and n^* from equations (28)-(29) after plugging the values of $a(.)$, $a_1(.)$ and $a_2(.)$, and then check if the solution is a local maximum and unique.

We can find an alternative solution by assuming that n(.,.) and s(.,.) are constant functions and then the system of equations that will produce this kind of steady-state subgame perfect gift equilibriuam is exactly the same as (28)-(30) with the exception that θ in the last bracketed term of (28) and 1 in the last bracketed term of (29) are omitted. We will see that that both types of steady-state subgame perfect gift equilibria exist with an example later.

In the rest of the paper, we study the properties of such steady-state subgame perfect gift equilibria.

Proposition 3 *Let v(.) be twice continuously differentiable with* $\mathrm{v}''(c) < 0\,\forall\, c > 0$*, then for all (n,s) that lead to positive consumption in each period, equation (30) has a continuously differentiable solution* $a(n,s)$ *and* $\partial a(n,s)/\partial s < 0$.

PROOF: For the implicit function $\Phi(n, s, a) = 0$ in (30), we have

$$\frac{\partial \Phi(.)}{\partial a} = -\mathrm{w}[\mathrm{v}''(c_1^*) + \mathrm{v}''(c_2^*)\delta(n)n^2/\alpha] > 0$$

Hence the first part follows from the implicit function theorem. Using the implicit function theorem again, we have

$$\frac{\partial a(n, s)}{\partial s} = -\frac{\mathrm{v}''(c_1^*) + (1 + r)\mathrm{v}''(c_2^*)\delta(n)n/\alpha}{w[\mathrm{v}''(c_1^*) + \mathrm{v}''(c_2^*)\delta(n)n^2/\alpha]} < 0$$

Q.E.D.

While the effect of parents savings is negative on the transfers from children, the correpsonding effect of number of children could be ambiguous. To show this, let us denote by $\Delta(n) \equiv \delta(n).n$ and assume that $\Delta(n)$ is an increasing function of n. Proceeding in the same manner as in the proof of above proposition, we can derive that

$$\frac{\partial a(.,.)}{\partial n} = -\frac{\Delta'(n)v'(c_2^*) + [\Delta(n)a(.,.)wv''(c_2^*) + \theta\alpha v''(c_1^*)]}{[w\alpha v''(c_1^*) + \Delta(n)wnv''(c_2^*)]}$$

Note that the bracketted terms in the above are negative and the first term of the numerator is positive. Thus sign of the right hand side of the above partial derive will depend on the relative magnitudes of the bracketted term and the first term on the numerator. In the example that we will consider later, the right hand side is unambiguously negative.

A steady-state open loop gift equilibrium is *manipulation proof* if agents do not have incentive to manipulate their children in order to extract more transfers from them.

Proposition 4 *Suppose a steady-state open loop gift equilibrium results in positive savings and strict inequality of the no bequest constraint (15), then the equilibrium is not manipulation proof and hence not Pareto Optimal.*

PROOF: Consider a steady-state open loop gift equilibrium, (n^*, s^*, a^*). Given the Inada condition on $\gamma(n)$ we know that $n^* > 0$. Let $s^* > 0$ and (15) be a strict inequality. Suppose (n^*, s^*, a^*) is manipulation proof. Then it is also a subgame perfect equilibrium of the second type satisfying (28)[without 1 in the last braketed term] as an equality, and thus we have

$$[-\alpha \text{v}'(c_1^*) + \beta \text{v}'(c_2^*)(1+r)] + w.a_2(n,s)\left\{\beta \text{v}'(c_2^*)n - \gamma(n)\text{v}'(c_1^*)\right\} = 0 \quad (31)$$

Since (n^*, s^*, a^*) and the associated c_1^* and c_2^* are open loop gift equilibrium, the first term under the square bracket in equation (31) is zero [cf. equation (16)]. By proposition 3, we have $a_2(.,.) < 0$. This implies that the term under the curly bracket in (31) is zero. But this contradicts the assumption that (15) is a strict inequality. This establishes the first part of the proposition.

Since agents are strictly better-off in the subgame perfect equilibrium, the open loop gift equilibrium of the proposition is not Pareto optimal.

Q.E.D.

It might seem that since steady-state subgame perfect equilibria with operative gifts are manipulation proof they are all Pareto optimal. This is not necessarily true as shown in the following proposition.

Proposition 5 *Consider an economy that has a steady-state subgame perfect gift equilibrium* $(s^*, n^*, a^*(.,.))$ *with* $s^* = 0$ *and no bequest constraint, (15), holds as a strict inequality, and suppose further that the equilibrium satisfies:*

$$\beta - \left(\frac{\gamma(n^*)}{n^*}\right) \cdot \left(\frac{\text{v}'(c_1^*)}{\text{v}'(c_2^*)}\right) \equiv \mu > 0 \textit{ and } \delta(n^*) < \mu$$

then all agents can be made better-off with a suitably designed pay-as-you-go social security program. Hence such an equilibrium is not Pareto optimal.

PROOF: Consider a pay-as-you-go social security program which marginally taxes all adult agents and redistributes the revenues equally among their old parents. Suppose for the moment that agents do not change their fertility and savings decisions in response to introduction of such a social security program. The utility gains of a representative agent is $n^* \beta \text{v}'(c_2^*)$ from the increased consumption in the old-age. The utility loss is given by $\alpha \text{v}'(c_1^*) + \gamma(n)\text{v}'(c_1^*)$, where the first term corresponds to welfare loss due to fall in own adult-age consumption and the second term corresponds to the welfare loss due to reduction in children's adult-age consumption. Thus the net gain is

$$\triangle U \;\; = \;\; n^* \beta \text{v}'(c_2^*) - \alpha \text{v}'(c_1^*) - \gamma(n^*)\text{v}'(c_1^*)$$

$$
\begin{aligned}
&= n^*\beta \mathrm{v}'(c_2^*) - \delta(n^*)n^*\mathrm{v}'(c_2^*) - \gamma(n^*)\mathrm{v}'(c_1^*) \\
&= n^*\left(\beta \mathrm{v}'(c_2^*) - \gamma(n^*)\mathrm{v}'(c_1^*)\right) - \delta(n^*)n^*\mathrm{v}'(c_2^*) \\
&> 0
\end{aligned}
$$

In deriving the above we have used equation (30) and the fact that equation (15) is a strict inequality by assumption.

It is clear that if the agents optimally adjust their fertility and savings decisions, the gains in utility will be even higher.

Q.E.D.

Social security not only can improve Pareto efficiency of a steady-state open loop gift equilibrium that is not manipulation proof, it can also improve Pareto efficiency of a steady-state locally subgame perfect gift equilibrium provided no bequest condition is a strict inequality. If the no-bequest condition is an equality, introduction of social security cannot improve Pareto efficiency.

3.2 The Example Continued

Let the utility function be a CEM function as in (17). For this utility function, we have the following explicit solution a(n,s) of equation (30):

$$
a(n,s) = \frac{(\delta_0/\alpha)^{1/\rho} n^{\delta_1/\rho}\left(w - [s+\theta n]\right) - (1+r)s}{w\left(n + (\delta_0/\alpha)^{1/\rho} n^{\delta_1/\rho}\right)} \tag{32}
$$

One can easily verify that both $a_1(.)$ and $a_2(.)$ are negative for this reaction function. We have shown that for this economy, and that each type of steady-state subgame perfect gift equilibrium exists. We will sketch the procedure to find the second type of steady-state subgame perfect gift equilibrium.

To determine the steady-state subgame perfect gift equilibrium, we know from proposition 4 that $s = 0$. The subgame perfect equilibrium number of children can be found from equation (29) when we substitute $s = 0$, using the above functional forms. Let us denote the resulting equation as h(n). The form h(n) is very complicated and we do not know its shape in general. We take the same parameter values as in example 2. The determination of n is shown in figure 2.[7] The subgame

[7] We have restricted the figure to a small neighborhood around the equilibrium, in which the curve looks linear.

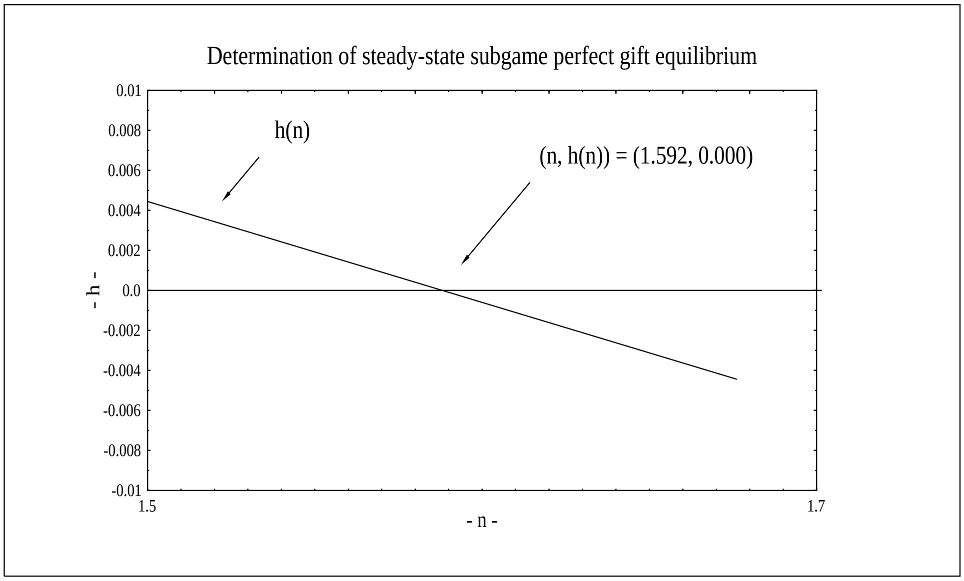

Figure 4: Determination of steady-state subgame gift perfect equilibrium

perfect gift equilibrium quantities are as follows: $(n, s, a) = (1.592, 0, .417)$ and $(c_1^*, c_2^*, U_{max}) = (5.668, 6.645, -1.15)$.

We also computed the first type of steady-state subgame perfect gift equilibrium as follows: $(n, s, a) = (1.51531, 0, .4232)$ and $(c_1^*, c_2^*, U_{max}) = (5.616, 6.413, -1.16)$.

3.3 PAYG Social Security

It is clear from propositions 3 and 4 that parents do have incentives to manipulate their children by consuming more in adult age, and saving nothing on physical assets and having more children to depend on for old-age support. By this manipulation they could receive a higher percentage of their children's income transferred to them, and assuming that their children will manipulate their children in the same way, everybody is made better-off in the subgame perfect equilibrium. If a pay-as-you-go social security program is introduced effecting the subgame perfect equilibrium transfers from children to parents, agents do not have incentive to manipulate their children's behavior to obtain this transfer and thus would save more on physical capital and have less children as a result of a publicly funded social security program. This predicted effect of the introduction of a social security is consistent with the stylized facts of many countries as reported in Nugent [1985].

The motive for social security in our view is to overcome the incentives to throw oneself to the mercy of the younger generation in old-age. Our view of social security is different from the social insurance view put forward by Diamond-Mirrlees [1978] and others. The purpose of social security is clearly more to force people to save for their retirement since we all know that we would not be able to let the elderly live miserably if they do not save for their retirement. Our view of social security is close to the social conscience view except that in our context the social conscience is extended to the family members only.

In our model, similarly to Veall [1986], social security benefits and taxes are endogenously determined. As in the Hansson and Stuart model, a social security tax-benefits stream for the current as well as all future generations that is implied by the subgame perfect gift equilibrium could be legislated by the living generations in period $t = 1$ and no future generations will have incentives to change it.

As such to attain the subgame perfect gift equilibrium allocation of proposition 2, it is not necessary to introduce a social security program. It is clear, however, from proposition 5 that if the no-bequest constraint is a strict inequality, such a subgame perfect equilibrium need not be Pareto optimal; Pareto optimality requires higher transfers from children to parents, and an appropriate PAYG social security program can serve such a social purpose.

4 Conclusion

In this paper we have considered a pure exchange overlapping generations model with two-sided limited altruism in the sense that agents care not only about their own life-cycle consumption, but they also care about their parents' old-age consumption and their children's adult-age consumption. In our economy agents decide their levels of fertility, savings, and transfers to parents and children. We compute open loop Nash equilibria as widely done in the literature. For a class of economies, we find that there are two steady-state open loop gift equilibria, one with positive savings and the other with zero savings; both equilibria coexist; moreover, the equilibrium with zero savings has higher fertility and utility levels of a representative agent in the steady-state. We then argue that an open loop Nash equilibrium ignores the sequential nature of the overlapping decision making of various generations and thus do not characterize the incentives that individuals face in their decisions.

A more appropriate framework is a sequential multi-stage game in extensive form, in which the notion of subgame perfect equilibrium is used to represent the equilibrium outcome of manipulation by parents. For the above class of economies, the steady-state subgame perfect equilibrium savings is always zero, and fertility and welfare levels are higher than in the open loop steady-state gift equilibrium with positive savings. We then argue that a PAYG social security program that sets benefits at the subgame perfect equilibrium levels of transfers can be legislated by the current living generations and no future generations will have incentives to amend it. However, if the no bequest constraint is a strict inequality, such a PAYG system does not lead to Pareto optimal allocations; Pareto optimality would require higher transfers from children to parents, and an appropriate PAYG social security program can serve such a social purpose.

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