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# Two-sided altruism, Lindahl equilibrium, and Pareto optimality in overlapping generations models*

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**Summary.** This paper extends the Samuelsonian overlapping generations general equilibrium framework to encompass a variety of altruistic preferences by recasting it into a Lindahl equilibrium framework. The First and the Second Welfare theorems hold for Lindahl equilibrium with respect to the Malinvaud optimality criterion but not with respect to the Pareto optimality criterion. A complete characterization of Pareto optimal allocations is provided using the Lindahl equilibrium prices.

**Keywords and Phrases:** Altruism, Lindahl equilibrium, Pareto optimality, Overlapping generations.

**JEL Classification Numbers:** D51, D62, D64, C62.

## 1 Introduction

Intergenerational altruism creates consumption externality. The Lindahl equilibrium is the analogue of the competitive equilibrium in the presence of externality and public goods. In this paper, I extend the Samuelsonian overlapping generations (OLG) general equilibrium framework to a Lindahl equilibrium framework that incorporates a variety of altruistic preferences, including two-sided paternalistic and non-paternalistic preferences, and preferences that are not necessarily inter-temporally consistent. Previous studies proved existence of Lindahl equilibrium for economies with finite number of commodities and agents (Foley [5], and Milleron [6]) and for economies with finite number of commodities and non-atomic

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measure space of agents of finite measure (Roberts [9]). These techniques do not apply to the present OLG set-up with a countably infinite number of agents and goods. For OLG models, Wilson [12] and Aliprantis, Brown and Burkinshaw [2] proved existence of competitive equilibrium assuming that agents are non-altruistic, and Aiyagari [1] proved existence of competitive equilibrium with voluntary bequest assuming that agents have time consistent non-paternalistic altruism towards their children. I adapt these results to a more general Lindahl equilibrium set-up that incorporates various types of altruism including the above.

For Arrow-Debreu economies with finite number of goods and agents, the First and the Second Welfare Theorems provide a complete characterization of the Pareto optimal allocations: An allocation is Pareto optimal if and only if it is a competitive equilibrium allocation with a redistribution of initial endowments. For OLG economies with non-altruistic agents, Samuelson [11] demonstrated the failure of these two theorems. Balasko and Shell [4] proved that these two welfare theorems hold with respect to the Weak Pareto Optimality criterion, also known as Malinvaud optimality criterion. They provided necessary and sufficient conditions for a competitive equilibrium to be Pareto Optimal using the competitive equilibrium prices. In this paper, I extend these results for OLG economies with altruistic agents.

The rest of the paper is organized as follows. Section 2 describes the basic framework. Section 3 deals with the existence of Lindahl equilibrium. Section 4 provides a complete characterization of Pareto optimal allocations in terms of Lindahl equilibrium prices.

## 2 Altruism and Lindahl equilibrium: The basic framework

For economies with finite number of agents and goods, the Lindahl equilibrium framework has been reformulated in various ways ([10,3,5], and [6]). For the OLG economies, I adapt the approach in [3] and reformulate the Lindahl equilibrium as the Walrasian equilibrium of an Arrow-Debreu economy on an extended set of goods.

I consider a standard OLG pure exchange economy with a stationary population in which one agent is born in each period $t \geq 0$, denoted as agent $t$. He lives only during periods $t$ and $t+1$. Let $A = \{0, 1, 2, \dots\}$ be the set of agents in the economy. Assume that there are $\ell$ perishable goods in each period. These goods are said to be *regular goods*. An agent $t \geq 1$ exhibits two-sided paternalistic altruism of the type that he derives utility from his own life time consumption and the consumption of the other family members that he can observe in his life-time, i.e., agent $t \geq 1$ has utility function, $u^t : \Re_+^{4\ell} \to \Re$, denoted by $u^t(x_t^{t-1}, x_t^t, x_{t+1}^t, x_{t+1}^{t+1})$. The first biological agent $t = 0$ has one sided altruism with the utility function, $u^0 : \Re_+^{3\ell} \to \Re$, denoted by $u^0(x_0^0, x_1^0, x_1^1)$. A bundle of goods is represented by a vector in $\Re^\infty$ with the convention that the first $\ell$ components of the vector correspond to the $\ell$ goods of period $t = 0$, followed by the next period's $\ell$ goods and so on. A *possible consumption bundle of regular goods* for agent $t$ is a vector, $x^t = (0, 0, \dots, 0, x_t^t, x_{t+1}^t, 0, \dots) \in \Re^\infty$, where $x_t^t, x_{t+1}^t \in \Re^\ell$. Let $\mathcal{X}^t$ be the set of all possible consumption bundles of agent t. Agent $t$ has an initial endowment $w^t = (0, 0, \dots, 0, w_t^t, w_{t+1}^t, 0, \dots) \in \Re_+^\infty$. The aggregate endowment is denoted by

$w = \Sigma_{\alpha \in A} w^{\alpha}$. Denote by $\mathbf{p} = (p_t)_0^{\infty} \in \Re^{\infty}$ the price vector whose indexing corresponds to the indexing of the consumption vectors in $\Re^{\infty}$.

A regular commodity that does not create externality is a *private good*, and all other goods whose consumption creates externality are *externality generating goods*. For each externality generating good $g$, and for each pair of agents involved as a server $s$ and as a receiver $r$ of externality of such a good, create two *externality goods rsg* and *ssg*. A commodity *lsg* is interpreted as agent *l*'s perception about agent *s*'s consumption of good *g*, for $l = r$ and $s$. The price of any externality good $lkt$ is denoted as $\pi_{lkt}$. The bundle of externality goods from agent $t$'s perspectives is denoted by $q_t = (q_{tt-1t}, q_{ttt}, q_{ttt+1}, q_{tt+1t+1})$ for $t \geq 1$ and by $q_0 = (q_{001}, q_{011})$ for $t = 0$. Notice that the components of $q_t$ are nothing but the consumption vector of regular goods, $(x_t^{t-1}, x_t^t, x_{t+1}^t, x_{t+1}^{t+1})$ in disguise. A bundle of externality goods is a vector $q = (q_0, q_1, \ldots ., q_t, \ldots) \in \Re_+^{\infty}$. *A possible consumption bundle of externality goods* for agent $t$ is a vector $q^t = (0, ..., 0, q_t, 0, ...) \in \Re^{\infty}, t \geq 0$. An *extended commodity bundle* is a vector $\tilde{x} = (x \mid q) \in \Re_+^{\infty}$ which is a bundle of regular goods $x$ and externality goods $q$. Initial endowment of extended goods for agent $\alpha, \alpha \in A$, is then $\tilde{w}^{\alpha} = (w^{\alpha} \mid 0)$. Denote the extended consumption set of agent $t$, $t \in A$ by $\tilde{\mathcal{X}}^t \subset \Re_+^{\infty}$. An *extended price vector* is a vector $\tilde{\mathbf{p}} = (\mathbf{p} \mid \pi) \in \Re^{\infty}$, where $\mathbf{p}$ is the price vector corresponding to the regular goods and $\pi$ is the price vector corresponding to externality goods. The *value* of an extended commodity bundle $\tilde{x}$ evaluated at a vector of extended prices $\tilde{\mathbf{p}}$ is defined as $\tilde{x}.\tilde{\mathbf{p}}$ $= \lim \ \inf_{n \to \infty} \sum_{t=1}^{n} (p_t.x_t + \pi_t.q_t)$.

For each agent $\alpha \in A$, a *preference ordering* $\succeq_{\alpha}$on $\tilde{\mathcal{X}}^{\alpha}$ is induced from his original utility function $u^{\alpha}$ on $\mathcal{X}^{\alpha}$ as follows: For agent $\alpha \geq 1$, define for any two vectors $\tilde{x} = (0|q^{\alpha})$ and $\tilde{x}^* = (0|q^{*\alpha}) \in \tilde{\mathcal{X}}^{\alpha}$ the preference ordering $\succeq_{\alpha}$by $\tilde{x} \succeq_{\alpha} \tilde{x}^* \iff u(q^{\alpha}) \geq u(q^{*\alpha})$. For agent $\alpha = 0$, define for any two vectors $\tilde{x} = (x^0|q^0)$ and $\tilde{x}^* = (x^{*0}|q^{*0}) \in \tilde{\mathcal{X}}^{\alpha}$ the preference ordering $\succeq_0$by $\tilde{x} \succeq_0$ $\tilde{x}^* \iff u(x^0, q^0) \geq u(x^{*0}, q^{*0})$. To each agent $t, t \in A$, assign an externality production possibility set in the extended commodity space $\Re^{\infty}$ as follows: For agent $t = 0$, define

$$\tilde{Y}^0 = \left\{ ((0, -x_1^0, 0 \ldots) \mid \left( (\overbrace{(x_1^0, 0)}^{q_0}, \overbrace{(x_1^0, 0, 0, 0)}^{q_1}, 0, \ldots) \right) \in \Re^{\infty} | x_1^0 \in \Re_+^{\ell} \right\}$$

and for any other agent $t \geq 1$, define

$$\tilde{Y}^t = \left\{ \left( (..., 0, -x_t^t, -x_{t+1}^t, 0...) \mid (.., 0, \overbrace{(0, 0, 0, x_t^t)}^{q_{t-1}}, \overbrace{(0, x_t^t, x_{t+1}^t, 0)}^{q_t}, \overbrace{(x_{t+1}^t, 0, 0, 0)}^{q_{t+1}}, 0, \ldots) \right) \in \Re^{\infty} \text{ such that } (x_t^t, x_{t+1}^t) \in \Re_+^{2\ell} \right\}$$

The interpretation of the production set $\tilde{Y}^t$ corresponding to agent $t$ is that he purchases $(x_t^t, x_{t+1}^t) \in \Re_+^{2\ell}$ from the regular goods markets and produces four output vectors - two extended goods $q_{ttt} = x_t^t$ and $q_{ttt+1} = x_{t+1}^t$ for his own consumption and the extended goods $q_{t-1tt} = x_t^t$ for externality consumption of agent $t-1$ and $q_{t+1tt+1} = x^t_{t+1}$ for the externality consumption of agent $t+1$. The aggregate externality production technology is defined by $\tilde{\mathbf{Y}} = \sum_{t=0}^{\infty} \tilde{Y}^t$. An *attainable allocation* is a collection $(\tilde{x}^\alpha, \tilde{y}^\alpha)_{\alpha \in A}$ such that $\tilde{x}^\alpha \in \tilde{\mathcal{X}}^\alpha$, and $\tilde{y}^\alpha \in \tilde{Y}^\alpha$ for all $\alpha \in A$ and $\sum_{\alpha \in A} \tilde{x}^\alpha = \sum_{\alpha \in A} (\tilde{y}^\alpha + \tilde{w}^\alpha)$.

**Definition 1** *A Lindahl Equilibrium for the economy* $\mathcal{E} = \left\langle \tilde{\mathcal{X}}^\alpha, \tilde{\mathbf{Y}}^\alpha, \tilde{w}^\alpha, \succeq_\alpha \right\rangle_{\alpha \in A}$ *is an extended price vector* $\tilde{\mathbf{p}}^* \in \Re^\infty_+$ *and an allocation* $(\tilde{x}^{\alpha *}, \tilde{y}^{\alpha *})_{\alpha \in A}$ *such that for each* $\alpha \in A$,

*(1)* $\tilde{\mathbf{p}}^*.(\tilde{x}^{\alpha *} - \tilde{w}^\alpha) \leq 0$ *and for any* $\tilde{x}^\alpha \in \tilde{\mathcal{X}}^\alpha$, $\tilde{x}^\alpha \succ_\alpha \tilde{x}^{\alpha *} \Rightarrow \tilde{\mathbf{p}}^*.\tilde{x}^\alpha > \tilde{\mathbf{p}}^*.\tilde{x}^{\alpha *}$
*(2)* $\tilde{\mathbf{p}}^*.\tilde{y}^{\alpha *} = 0$ *and* $\tilde{\mathbf{p}}^*.\tilde{y}^\alpha \leq 0$ *for all* $\tilde{y}^\alpha \in \tilde{Y}^\alpha$
*(3)* $\sum_{\alpha=0}^\infty \tilde{x}^{\alpha *} = \sum_{\alpha=0}^\infty \tilde{y}^{\alpha *} + \sum_{\alpha=0}^\infty \tilde{w}^\alpha$.

Applying condition (2) of the above definition to $\alpha = t$ and $\alpha = t-1$, it is clear that the Lindahl equilibrium prices satisfy,

$$\forall t \geq 1, \quad \begin{cases} p_t = \pi_{t-1tt} + \pi_{ttt} \\ p_t = \pi_{t-1t-1t} + \pi_{tt-1t} \end{cases} \tag{1}$$

The first equation in (1) says that the public price $p_t$ of the private goods in period $t$ equals the sum of the private prices $\pi_{t-1tt}$ and $\pi_{ttt}$ of the public goods. Similar is the interpretation of the second equation.

## 3 The existence of Lindahl equilibrium

I adapt the proof of the existence theorem of Wilson [12] for this extended OLG set-up with joint production and no free disposal. I construct the following sequence of finite subeconomies, $\mathcal{E}_n = \left\langle \tilde{\mathcal{X}}^\alpha_n, \tilde{\mathcal{Y}}^\alpha_n, \tilde{w}^\alpha_n, \succeq^n_\alpha \right\rangle, \alpha \in A_n, n \geq 0$, involving finite dimensional commodity spaces with goods indexed by $C_n$ and finite number of agents indexed by $A_n$, where $C_0 = \{0, 1, 001\}, C_1 = C_0 \cup \{2, 011, 101, 111, 112\}$, ..., $C_t = C_{t-1} \cup \{t+1, t-1tt, tt-1t, ttt, ttt+1\}$, for all $t \geq 1$ and $A_0 = \{0\}$, $A_1 = \{0, 1\}$, ..., $A_n = \{0, 1, \ldots, n\}$, for all $n \geq 0$. For any extended commodity $\tilde{x}$, define a sequence of truncated extended commodity vectors $\tilde{x}_n$, $n \geq 0$ by assigning the same numbers as in $\tilde{x}$ corresponding to the components in $C_n$and zeros for the remaining components. For $n \geq 0$ and $\alpha \in A_n$, define $\tilde{\mathcal{X}}^\alpha_n, \tilde{\mathcal{Y}}^\alpha_n$ by $\tilde{\mathcal{X}}^\alpha_n = \{\tilde{x}_n \in \Re^\infty_+ \mid$the associated $\tilde{x} \in \tilde{\mathcal{X}}^\alpha\}$ and $\tilde{\mathcal{Y}}^\alpha_n = \{\tilde{y}_n \in \Re^\infty_+ \mid$the associated $\tilde{y} \in \tilde{\mathcal{Y}}^\alpha\}$. Define a preference ordering $\succeq^n_\alpha$on $\tilde{\mathcal{X}}^\alpha_n$ as follows: for any two $\tilde{x}^\alpha_n, \tilde{x}^{\alpha *}_n \in \tilde{\mathcal{X}}^\alpha_n$, define $\succeq^n_\alpha$by $\tilde{x}^\alpha_n \succeq^n_\alpha \tilde{x}^{\alpha *}_n \iff u^\alpha(q^\alpha) \geq u^\alpha(q^{\alpha *})$ if $0 < \alpha < n$, $u^\alpha(q^\alpha, w^\alpha_{n+1}) \geq u^\alpha(q^{\alpha *}, w^\alpha_{n+1})$ if $\alpha = n$, and $u^\alpha(x_0, q_0) \geq u^\alpha(x^*_0, q^*_0)$ if $\alpha = 0$.

I modify the irreducibility and resource relatedness concepts in [2,7], and [12] as follows: For any $B \subset A$, denote by $\tilde{w}_B = \Sigma_{\alpha \in B} \tilde{w}_\alpha$. Suppose an economy consists of a set of finite or infinite number of agents A, an aggregate production set $\tilde{\mathbf{Y}}$ and an aggregate endowment of the extended goods $\tilde{w}$. The economy is said to be *irreducible* if for any partition $A_1$ and $A_2$ of $A$, (i.e., $A_1, A_2 \neq \phi, A_1 \cup A_2 = A$) and for any consumption allocation $(\tilde{x}^\alpha)_{\alpha \in A}$ such that $\sum_{\alpha \in A} \tilde{x}^\alpha = \tilde{y} + \tilde{w}$, $\tilde{y} \in \tilde{\mathbf{Y}}, \exists \alpha' \in A_1$ and a finite set $B' \subset A_2, 0 \leq \bar{w} \leq \tilde{w}_{B'}$ such that $\bar{w} + \tilde{x}^{\alpha'} \succ_{\alpha'} \tilde{x}^\alpha$.

**Theorem 1** *Assume that (1) the aggregate endowment* $w \gg 0$, *(2) the preference order* $\succeq_\alpha$ *is continuous with respect to the product topology on* $\tilde{\mathcal{X}}^\alpha$, *weakly monotonic, and convex, for each* $\alpha, \alpha \in A$, *and (3) the full economy* $\mathcal{E}$ *and all finite subeconomies* $\mathcal{E}_n, n \geq 0$ *are irreducible. Then there exists a Lindahl equilibrium for* $\mathcal{E}$.

*Proof.* The sequence of finite economies $\mathcal{E}_n$, $n \geq 0$ satisfy all the assumptions of McKenzie ([7], Theorem 1, pp.828). Hence there exists a Lindahl equilibrium $\langle \tilde{\mathbf{p}}_n, (\tilde{x}_n^\alpha, \tilde{y}_n^\alpha)_{\alpha \in A_n} \rangle$ for each $\mathcal{E}_n$. The weak monotonicity of preference orderings guarantees that $\tilde{\mathbf{p}}_n > \mathbf{0}$. I want to show that there exists a subsequence of economies for which $\langle \tilde{\mathbf{p}}_n, (\tilde{x}_n^\alpha, \tilde{y}_n^\alpha)_{\alpha \in A_n} \rangle \longrightarrow \langle \tilde{\mathbf{p}}^*, (\tilde{x}^{\alpha *}, \tilde{y}^{\alpha *})_{\alpha \in A} \rangle$ in product topology and that $\langle \tilde{\mathbf{p}}^*, (\tilde{x}^{\alpha *}, \tilde{y}^{\alpha *})_{\alpha \in A} \rangle$ is a Lindahl equilibrium for the full economy $\mathcal{E}$.

To that end, I first show that there exist bounds $M^\alpha \in \Re_+^\infty$ independent of $n$ such that $0 \leq \tilde{x}_n^\alpha \leq M^\alpha$ and $-M^\alpha \leq \tilde{y}_n^\alpha \leq M^\alpha$ for all $\alpha \in A$. For, let $M^\alpha$ be the vector in the extended commodity space such that its each component involving $t$-th period goods consists of $w_t \in \Re_+^\ell$, the aggregate endowment of $t$-th period goods in the economy. Since each unit of the externality good of any physical characteristics, available at any time and defined for any two agents is produced by one and only one production plan which uses one unit of the regular good of the same characteristics and available in the same period, therefore, in each finite sub-economy they ought to be bounded by the aggregate endowment of that good. Since the sequence $\{\tilde{x}_n^\alpha, \tilde{y}_n^\alpha\}_{n \geq 0}$ is uniformly bounded for each $\alpha$, there exists a subsequence $\{n_k\}$ such that as $k \to \infty$, $\tilde{x}_{n_k}^\alpha \to \tilde{x}^{\alpha *}$, and $\tilde{y}_{n_k}^\alpha \to \tilde{y}^{\alpha *}$ in the product topology. Restrict the rest of the proof to this subsequence and use index $n$ instead of $n_k$ to denote it.

Next I show that $\tilde{p}_n$ is uniformly bounded above and below, and hence, there is a subsequence of economies for which $\tilde{p}_n \longrightarrow \tilde{p}^*$ in product topology. To that end, normalize prices $\tilde{p}_n$ such that $\tilde{p}_n.\tilde{w}_n^0 = 1$ for all $n \geq 0$. Mimicking the proof of lemma 3 in [12], it can be shown that for any $\beta, \gamma \in A, \exists K_{\beta\gamma} > 0$ such that $0 < \tilde{\mathbf{p}}_n.\tilde{w}^\gamma < K_{\beta\gamma} \cdot \tilde{\mathbf{p}}_n.\tilde{w}^\beta$ for all $n$ such that $\beta, \gamma \in A_n$. Since $\mathcal{E}_n$ is irreducible, for any $n$, any time $t \leq n$, and any regular good, $\exists \alpha \in A_n$ who has a positive initial endowment of that good. Take $\gamma$ to be such an $\alpha$ and $\beta$ to be 0. Thus, we have $0 < \tilde{\mathbf{p}}_n.\tilde{w}^\alpha < K_{\alpha 0}$. From this it follows that the price vector of regular goods $p_n$ is uniformly bounded. But equation (1) tells us that the prices of the externality goods are bounded by the prices of the regular goods, hence it follows that $\{\tilde{p}_n\}_0^\infty$ is uniformly bounded.

I now show that $\langle \tilde{p}^*, (\tilde{x}^{\alpha *}, \tilde{y}^{\alpha *})_{\alpha \in A} \rangle$ is a Lindahl equilibrium for the economy $\mathcal{E}$. To that end, note that for any $\alpha \in A, \tilde{p}_n.\tilde{y}_n^\alpha = 0 \Rightarrow \lim_{n \to \infty} \tilde{p}_n.\tilde{y}_n^\alpha = \tilde{p}^*.\tilde{y}^{\alpha *} = 0$. Similarly, $\tilde{p}^*.\tilde{y}^\alpha \leq 0$ for any $\tilde{y}^\alpha \in \tilde{Y}^\alpha$. This establishes condition (2) in the definition of Lindahl equilibrium. Finally, note that $\tilde{p}_n.\tilde{x}_n^\alpha = \tilde{p}_n.\tilde{w}^\alpha$ for large $n$. Therefore, $\tilde{p}^*.\tilde{x}^{\alpha *} = \tilde{p}^*.\tilde{w}^\alpha$. Next I show that $\tilde{x}^\alpha \succ_\alpha \tilde{x}^{\alpha *} \Rightarrow \tilde{p}^*.\tilde{x}^\alpha > \tilde{p}^*.\tilde{x}^{\alpha *}$.

*Case 1.* Suppose $\tilde{p}^*.\tilde{x}^\alpha < \tilde{p}^*.\tilde{x}^{\alpha *}$. Then by the continuity of $\succeq_\alpha$, for some large $n, \tilde{x}_n^\alpha \succ_\alpha^n \tilde{x}_n^\alpha$ and $\tilde{p}_n.\tilde{x}_n^\alpha < \tilde{p}_n.\tilde{x}_n^\alpha$. This is a contradiction.

*Case 2.* Suppose $\tilde{p}^*.\tilde{x}^\alpha = \tilde{p}^*.\tilde{x}^{\alpha *}$. The continuity of $\succeq_\alpha$ implies that $\lambda \tilde{x}^\alpha \succ_\alpha \tilde{x}^{\alpha *}$, for some $\lambda < 1$. Therefore, again by continuity of $\succeq_\alpha$, for some large $n$, $\lambda \tilde{x}_n^\alpha \succ_\alpha^n \tilde{x}_n^\alpha$

with $\tilde{p}_n.\lambda\tilde{x}_n^\alpha < \tilde{p}_n.\tilde{x}_n^\alpha$. (Suppose, instead, that $\tilde{p}_n.\lambda\tilde{x}_n^\alpha \geq \tilde{p}_n.\tilde{x}_n^\alpha$ for all $n$, then taking the limit as $n \to \infty$ on both sides, we would get $\lambda\tilde{p}^*.\tilde{x}^\alpha = \tilde{p}^*.\tilde{x}^{\alpha*}$. That is incredible). But then $(\tilde{x}_n^\alpha)_{\alpha\, \in\, A}$ could not be an equilibrium for $\mathcal{E}_n$. This verifies condition (1) in the definition of Lindahl equilibrium.

The condition $\sum_{\alpha\in A_n}\tilde{x}^{\alpha*} = \sum_{\alpha\in A_n}\tilde{y}^{\alpha*} + \sum_{\alpha\in A_n}\tilde{w}^\alpha$ holds for each subeconomy $\mathcal{E}_n$. Hence in the limit, it holds for the full economy $\mathcal{E}$. This establishes condition (3). □

## 4 The Lindahl equilibrium and the Pareto optimality

An attainable allocation $(\tilde{x}^\alpha,\ \tilde{y}^\alpha)_{\alpha\in A}$ in the extended commodity space is *Pareto optimal* if there does not exist another attainable allocation $(\tilde{x}'^\alpha,\ \tilde{y}'^\alpha)_{\alpha\in A}$ such that $\tilde{x}'^\alpha \succeq_\alpha \tilde{x}^\alpha$ with strict preference for some $\alpha \in$ A. It is *Weakly Pareto Optimal* or *Malinvaud Optimal* if there does not exist another attainable allocation $(\tilde{x}'^\alpha,\ \tilde{y}'^\alpha)_{\alpha\in A}$ and a $\tau \geq 1$ such that $\tilde{x}^t = \tilde{x}'^t$ and $\tilde{y}^t = \tilde{y}'^t$ for all $t \geq \tau$ and $\tilde{x}'^t \succeq_\alpha \tilde{x}^t$ for all $t \geq 0$ with strict preference for at least one t. The corresponding definitions on the regular commodity space are straightforward. Following the line of arguments in [4], it is easy to prove the following theorem:

**Theorem 2** *A Lindahl equilibrium is Malinvaud Optimal and any Malinvaud optimal allocation can be supported by a vector of Lindahl equilibrium prices.*

A Pareto optimal allocation is always a Lindahl equilibrium allocation. When is a Lindahl equilibrium allocation also a Pareto optimal allocation? I restrict my exposition to the case $\ell = 1$. Assume that all utility functions are $C^2$-smooth. Fix a Lindahl equilibrium allocation $\tilde{x} = (x|q)$ and $\tilde{p} = (p|\pi)$ . An intergenerational *transfer scheme* is a vector $h = (h_1, ...h_t, ...), h_t \in \Re$, where $h_t$ is the amount of the t-th period good taxed on the adult agent $t$ and given as a gift to the old agent *t-1* in period $t$. For any transfer scheme $h = (h_1, h_2, ...h_t, ...)$ , denote by $\left(\sigma_t^t, \sigma_{t+1}^t\right) = (-h_t, h_{t+1})$ the vector of transfers to agent $t$ from agent $t-1$ and agent $t+1$ respectively. A transfer scheme $h = (h_1, ...h_t)$ is *feasible* if $x_t^t - h_t \geq 0$ for positive $h_t$ and $x_t^{t-1} + h_t \geq 0$ for negative $h_t$, for all $t \geq 1$, and is *Pareto improving* if it is feasible and $U^0(x_1^0 + h_1 x_1^1 - h_t) \geq \bar{U}^0$, $U^t(x_t^{t-1} + h_t, x_t^t - h_t, x_{t+1}^t + h_{t+1}, x_{t+1}^{t+1} - h_{t+1}) \geq \bar{U}^t$, for all $t \geq 1$ with strict inequality for some $t$, where $\bar{U}^t$ denotes the utility level of agent $t \geq 0$ at the fixed Lindahl equilibrium. Denote by $\breve{H}^t$ the set of transfers $\left(\sigma_t^t, \sigma_{t+1}^t\right) \in \Re^2$ to agent $t$ such that $U^t(x_t^{t-1} - \sigma_t^t, x_t^t + \sigma_t^t, x_{t+1}^t + \sigma_{t+1}^t, x_{t+1}^{t+1} - \sigma_{t+1}^t) \geq \bar{U}^t$.

An intergenerational transfer of a unit of regular good of period $t$ to agent $t$ from agent $t-1$ corresponds to a unit increase in his consumption of good $ttt$ and a unit decrease in the consumption of good $tt-1t$. The personalized unit price of such a transfer to agent $t$ is $\phi_t \equiv \pi_{ttt} - \pi_{tt-1t}$. Similarly, the personalized price to agent $t$ per unit transfer of good from agent *t+1* to agent $t$ is given by $\pi_{ttt+1} - \pi_{tt+1t+1}$. Given a transfer scheme $h = (h_1, h_2, ...h_t, ...)$ , denote by $\mu_t \equiv \phi_{t+1}h_{t+1} - \phi_t h_t$ and by $\eta_t = 1$ if $(-h_t, h_{t+1}) \in \breve{H}^t$ and $\eta_t = \sup\left\{\lambda > 0 | (-\lambda h_t, \lambda h_{t+1}) \in \breve{H}^t\right\}$ otherwise. The necessary and sufficient conditions for Pareto optimality are stated in the following theorem:

**Theorem 3** *Let $x = \left(x^0, x^1, ...x^t, ...\right)$ be an allocation of regular goods corresponding to a Lindahl equilibrium allocation of extended goods, $\tilde{x} = \left(\tilde{x}^0, \tilde{x}^1, ..., \tilde{x}^t, ...\right)$, and extended prices, $\tilde{\mathbf{p}} = (p|\pi)$ such that*

*(a)* $\phi_t \equiv \pi_{ttt} - \pi_{tt-1t} > 0 \, \forall t \geq 1$

*(b) curvatures of the indifference surfaces of all agents passing through the Lindahl equilibrium allocation are such that for any sequence $h = (h_1, h_2, ...)$, $0 < h_t \leq x_t^t$, $t \geq 1$ the associated sequence $\{\eta_t\}_1^\infty$ is uniformly bounded away from below, i.e., there exists $\underline{\eta}, 1 > \underline{\eta} > 0$ such that $\eta_t \geq \underline{\eta}$ for all $t \geq 1$.*

*(c) the Gaussian curvature of indifference curve passing through the Lindahl equilibrium allocation at any transfer $\left(\sigma_t^t, \sigma_{t+1}^t\right) = (-h_t, h_{t+1})$ associated with a feasible transfer scheme $h = (h_1, h_2, ...)$ is uniformly bounded away from 0.*

*(d)* $\left(x_t^t, x_{t+1}^t\right)$ *is uniformly bounded away from below and above.*

*Then, $\tilde{x}$ is Pareto optimal* if and only if $\sum_{t=1}^{\infty} \frac{1}{\phi_t} = \infty$.

*Proof.* I first show that if $\sum_{t=1}^{\infty} \frac{1}{\phi_t} < \infty$, there exists a feasible transfers program $h = (h_1, h_2, ...)$ which is Pareto improving upon $x$. From the definition of $\mu_t$ it follows that $\phi_{t+1} h_{t+1} = \phi_t h_t + \mu_t = \phi_1 h_1 + \sum_{\tau=1}^{t} \mu_\tau$. Taking $\mu_t = \frac{1}{\phi_t}$,we have $h_{t+1} = \frac{\phi_1 h_1}{\phi_{t+1}} + \frac{1}{\phi_{t+1}} \sum_{\tau=1}^{t} \frac{1}{\phi_\tau} \leq \frac{\phi_1 h_1}{\phi_{t+1}} + \frac{1}{\phi_{t+1}} \sum_{\tau=1}^{\infty} \frac{1}{\phi_\tau}$. Note that $\sum_{\tau=1}^{\infty} \frac{1}{\phi_\tau} < \infty$ $\Rightarrow \frac{1}{\phi_t} \to 0$ as $t \to \infty \Rightarrow h_{t+1} \to 0$ as $t \to \infty$. Hence, $h_t$ is bounded above and is $> 0 \; \forall t \geq 1$. If $h = (h_1, h_2, ...)$ is not feasible, we can always find a multiplier $\gamma, 0 < \gamma < 1$, such that $\hat{h} = (\gamma h_1, \gamma h_2, ...)$ is feasible and bounded from above. Consider the feasible transfer scheme $\breve{h} = (\breve{h}_1, \breve{h}_2, ...)$ where $\breve{h}_t = \underline{\eta} \gamma h_t$, $t \geq 1$, and $\underline{\eta}$ is as in assumption (b). Utilizing equation (1), it is easy to see that $\pi_{001} - \pi_{011} = \phi_1$ and it is positive by assumption (a). Since $\pi_{001} > \pi_{011}$, it follows that a transfer $\breve{h}_1 > 0$ from an adult to old in period $t = 1$ will increase the utility of agent $t = 0$, and for all other agents $t \geq 1$ the associated feasible transfers $\left(\sigma_t^t, \sigma_{t+1}^t\right) = \left(-\breve{h}_t, \breve{h}_{t+1}\right)$ will guarantee that they are not worse-off compared to the given Lindahl equilibrium. Hence the Lindahl equilibrium allocation is not Pareto Optimal.

I now prove the converse, i.e., if a Lindahl equilibrium allocation $\tilde{x}$ is not Pareto Optimal, then $\sum_{\tau=1}^{\infty} \frac{1}{\phi_\tau} < \infty$. Since $\tilde{x}$ is not Pareto optimal, there exists a feasible transfer scheme $h = (h_1, h_2, ..)$ which is Pareto improving upon $\tilde{x}$. Since $\pi_{ttt+1} - \pi_{tt+1t+1} = \phi_{t+1} > 0$ (by assumption (a)), a transfer from agent *t* to agent *t+1* at the fixed Lindahl equilibrium reduces agent $t$'s utility. Thus, the first non-zero component of a Pareto improving transfer scheme must have a positive element and it can be seen that all other components thereafter are also strictly positive. Without loss of generality, assume that $h_1 > 0$. Define $\alpha_t = \left(h_t^2 + h_{t+1}^2\right) \cdot \left(\phi_t^2 + \phi_{t+1}^2\right)^{1/2} / \mu_t$. Using property (c) and applying lemma 5.7 of [4], we see that $\alpha_t$ is uniformly bounded from above by a constant $K > 0$. Hence, $K > \alpha_t = \left(h_t^2 + h_{t+1}^2\right) \cdot \left(\phi_t^2 + \phi_{t+1}^2\right)^{1/2} / \mu_t = \phi_{t+1} h_{t+1}^2 \left(1 + h_t^2 / h_{t+1}^2\right) \left(1 + \phi_t^2 / \phi_{t+1}^2\right)^{1/2} / \mu_t >$ $\phi_{t+1} h_{t+1}^2 / \mu_t > \left(\mu_1 + ... + \mu_t\right)^2 / \phi_{t+1} \mu_t$. Thus from the last inequality of the above, we have $\frac{1}{\phi_{t+1}} < K \frac{\mu_t}{(\mu_1 + ... + \mu_t)^2}$. In the proof of lemma 5.10 in [4], it is shown that $\sum_{t=1}^{\infty} \frac{\mu_t}{(\mu_1+\ldots+\mu_t)^2} < \infty$ for any positive sequence of real numbers $\mu_t$, $t \geq 1$. Using this result, we have $\sum_{t=1}^{\infty} \frac{1}{\phi_t} < \infty$. □

*Remark 1* Theorem 3 could be easily extended to $\ell > 1$. Balasko and Shell [4] imposed a uniform Gaussian curvature restriction which implies property (b) in Theorem 3. Other properties of the equilibrium can be found in Raut [8]. The above results extend easily to other types of utility functions, including non-paternalistic two-sided altruistic preferences of the form: $u^t(x_t^{t-1}, x_t^t, x_{t+1}^t, x_{t+1}^{t+1}, x_{t+2}^{t+1}, x_{t+2}^{t+2}, \ldots)$ for all $t \geq 1$, and $u^t(x_t^t, x_{t+1}^t, x_{t+1}^{t+1}, \ldots)$ for $t = 0$. Notice that for the existence of equilibrium, I did not require any intertemporal consistency condition which was imposed for bequest equilibrium by Aiyagari. It will be interesting to study the role of Lindahl equilibrium prices in designing optimal social policies in the presence of consumption externality and various types of public goods along the lines discussed in Milleron [6].

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