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Dynamics of endogenous growth*

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Summary. We present an overlapping generations model of endogenous fertility and growth. The cost of child rearing and the effect of population size on total factor productivity determine the dynamics of competitive equilibrium path of our model. The non-linear dynamics of the model generates a plethora of outcomes (depending on the functional forms, parameters and initial conditions) that include not only the neo-classical steady state with exponential growth of population with constant per capita income and consumption, but also growth paths which do not converge to a steady state and are even chaotic. Exponential, and even super exponential, growth of per capita output are possible in some cases.

1. Introduction

The consequences of private reproduction and capital (physical and human) accumulation decisions to long-run economic development have been the focus of research of a number of scholars in recent years (National Research Council [1986], Nerlove [1987], and Raut [1985, 1991]). In the earlier literature on growth and development, household formation, schooling, fertility and labor force participation decisions of households, their mortality experience and the resulting rate of population and labor force growth were assumed to be exogenous. The recent literature, in contrast, explicitly recognizes their endogeneity. In addition, greater emphasis is placed on human as contrasted with physical capital in the growth process. Another strand of recent literature (Lucas [1988] and Romer [1986]) besides endogenizing growth, obtains sustained growth in per worker output and consumption, primarily by generating increasing scale economies in aggregate production.

Almost all recent contributions to endogenous growth theory focus on models for which an equilibrium steady state growth path exists, although it need not be unique or independent of initial condition. Analytical attention is focused almost

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exclusively on the properties of steady-state paths. In contrast, our model is not geared to generating balanced growth. In fact, the non-linear dynamics of the model generates a plethora of outcomes (depending on the functional forms, parameters and initial conditions) that include not only the neo-classical steady state with exponential growth of population with constant per capita income and consumption, but also growth paths which do not converge to a steady state and are even chaotic. Per capita output grows exponentially (and super exponentially) in some economies.

In section 2, we present our basic overlapping generations economy and derive a system of difference equations whose solution yields the equilibrium path of the underlying economy. Dynamic properties of the equilibrium path in our model depends on the nature of child rearing cost and the way population size affects total factor productivity. In section 3, assuming child rearing cost to be constant over time, we characterize the equilibrium paths of our economy as a solution of one-dimensional iterative maps, and study the complex dynamics of such a system. In section 4, we analyze dynamics of economies with more general child rearing cost functions.

2. Basic model

The economy produces a single commodity which can be consumed or accumulated. The production function for this commodity is given by

$$Y_t = G(L_t)F(K_t, L_t) \tag{1}$$

where K_t is the stock of capital and L_t is the size of working population in period t; $G(L_t)$ represents Hicks-neutral total factor productivity that depends on the size of the working population.

Assumption A:1. *F is a concave, homogeneous of degree 1 function of class C^2 and satisfies Inada condition:*

$$\lim_{k\to\infty} f'(k) = 0, \quad \textit{and} \quad \lim_{k\to\infty} f'(k) = \infty$$

where $f(k) \equiv F(k, 1)$.

Production is organized in competitive firms which maximize profit in each period given factor prices. Under assumption A:1 and with $G(L_t)$ as an externality this implies that

$$w_t = G(L_t)[f(k_t) - k_t f'(k_t)] \equiv G(L_t)w(k_t) \tag{2}$$

$$q_t = G(L_t)f'(k_t) \equiv G(L_t)q(k_t) \tag{3}$$

where, w_t is the wage rate and q_t is rental per unit of capital in period t, both in units of output.

At each t there are three overlapping generations of individuals. A typical member of each generation lives for three periods, the first of which is spent as a child in the parent's household. The second period is spent as a young person working, having and raising children, as well as accumulating capital. The third and last period of life is spent as an old person in retirement living off support received from each of one's offspring and from the sale of accumulated capital. All members of each generation have identical preferences defined over their consumption in their working and retired periods. Thus, in this model the only reason that an individual would want to have a child is the support the child will provide during her retirement. Firms buy capital from the retired and hire the young as workers. Markets for product, labor and capital are assumed to be competitive.

Formally, a typical individual of the generation which is young in period t has n_t children (reproduction is by parthenogenesis!), consumes c_t^t, c_{t+1}^t in periods t and $t+1$, and saves s_t in period t. She supplies one unit of labor for wage employment. Her income from wage labor while young in period t is w_t and that is the only income in that period. A proportion a of this wage income is given to parents as old-age support. While old in period $t+1$, she sells her accumulated saving to firms and receives from each of her offspring the proportion a of his/her wage income. She enjoys a utility $U(c_t^t, c_{t+1}^t)$ from consumption. Thus her choice problem can be stated as:

$$\max_{s_t, n_t > 0} U(c_t^t, c_{t+1}^t)$$

$$c_t^t + \theta_t n_t + s_t = (1-a)w_t \tag{4}$$

$$c_{t+1}^t = q_{t+1} s_t + a w_{t+1} n_t \tag{5}$$

First order necessary conditions of the above optimization problem are as follows:

$$U_1(c_t^t, c_{t+1}^t) \leq \lambda_t \quad \text{with equality if} \quad c_t^t > 0 \tag{6}$$

$$U_2(c_t^t, c_{t+1}^t) \leq \lambda_{t+1} \quad \text{with equality if} \quad c_{t+1}^t > 0 \tag{7}$$

$$-\lambda_t \theta_t + \lambda_{t+1} a w_{t+1} \leq 0 \quad \text{with equality if} \quad n_t > 0 \tag{8}$$

$$-\lambda_t + \lambda_{t+1} q_{t+1} \leq 0 \quad \text{with equality if} \quad s_t > 0 \tag{9}$$

where λ_t and λ_{t+1} are the Lagrange multipliers corresponding to the constraints (4) and (5) respectively. The analysis below assumes that there is an interior maximum in the sense that $c_t^t > 0$, $c_{t+1}^t > 0$, $n_t > 0$ and $s_t > 0$. Let L_t be the number of adults in period t, and K_t be the stock of capital in the beginning of period t. We assume that capital depreciates in one period. Thus we have the following macro relationships:

$$K_{t+1} = L_t s_t \quad \text{and} \quad L_{t+1} = L_t n_t \tag{10}$$

Let the initial capital and labor at $t=0$ be given as K_0 and L_0. A *competitive equilibrium path* of the economy $\mathscr{E} = \langle U, f, G \rangle$ is a sequence $\{s_t, n_t, c_t^t, c_{t+1}^t, L_{t+1}, K_{t+1}, q_t, w_t, \lambda_t\}_0^\infty$ such that equations (2)–(3) and (6)–(10) are satisfied.

Assumption A:2. *U is a strictly quasi-concave C^2 function and*

$$\lim_{c_1 \to 0} U_1(c_1, c_2) = \infty \ \forall c_2 > 0 \quad \textit{and} \quad \lim_{c_2 \to 0} U_2(c_1, c_2) = \infty \ \forall c_1 > 0$$

Assumptions A:1 and A:2 imply that in a competitive equilibrium $K_{t+1}, L_{t+1} > 0$, for all $t \geq 0$. Equations (8) and (9) imply that $q_{t+1} = \frac{aw_{t+1}}{\theta_t}$. Using equations (2)–(3), the following holds in competitive equilibrium:

$$\frac{f(k_{t+1}) - k_{t+1}f'(k_{t+1})}{f'(k_{t+1})} = \frac{\theta_t}{a}. \tag{11}$$

Lemma 1. *Under assumption A:1, equation (11) has a unique solution* $k_{t+1} = \Psi(\theta_t/a)$.

Proof: See Raut [1992].

In order to find the rest of the equilibrium quantities, we note that $k_{t+1} = \frac{s_t}{n_t}$ which holds under the assumption that capital depreciates fully in one generation. Substituting the solution $\Psi(\theta_t/a)$ for k_{t+1}, we get $s_t = n_t \Psi(\theta_t/a)$. Defining $\theta_t n_t + s_t = [\theta_t + \Psi(\theta_t/a)]n_t$ as S_t and noting that $aw_{t+1} = \theta_t q_{t+1}$, the budget constraints (4) and (5) can be written as $c_t^t = (1-a)w_t - S_t$ and $c_{t+1}^t = q_{t+1}S_t$. Substituting these in (6) and (7) (which are equalities at an interior maximum) and noting that $q_{t+1} = \frac{\lambda_t}{\lambda_{t+1}}$ from (9) we get

$$-U_1[(1-a)w_t - S_t, q_{t+1}S_t] + q_{t+1}U_2[(1-a)w_t - S_t, q_{t+1}S_t] = 0 \tag{12}$$

Lemma 2. *Under assumption A:2, there is a unique solution* $S_t = H(w_t, q_{t+1})$ *to (12).*

Given equilibrium $\Psi(a_t/\theta)$ and $H(\cdot)$, the equilibrium n_t and s_t are computed as follows:

$$n_t = \frac{H(w_t, q_{t+1})}{\theta_t + \Psi(\theta_t/a)}; \quad \text{and} \quad s_t = \frac{H(w_t, q_{t+1})}{\Psi(\theta_t/a)[\theta_t + \Psi(\theta_t/a)]}$$

In what follows, we further simplify our exposition by restricting utility functions to the following Cobb–Douglas class:

$$U(c_t^t, c_{t+1}^t) = \delta \log c_t^t + (1-\delta)\log c_{t+1}^t, \quad 0 < \delta < 1 \tag{13}$$

Under the above assumption,

$$H(w_t, q_{t+1}) = (1-\delta)(1-a)w_t, \quad \text{and} \quad n_t = \frac{(1-\delta)(1-a)w_t}{\theta_t + \Psi(\theta_t/a)},$$

which does not depend on q_{t+1}. We also assume that

$$\theta_t = \theta + \gamma G(L_t) + \rho w_t \tag{14}$$

In the above specification of the child rearing cost function, θ represents a time-invariant cost of child rearing to be thought of as the cost of material (i.e., commodity) cost. We allow the factors that bring about positive (or negative) externalities associated with population density (e.g., congestion or economies of scale in schooling) to influence the cost of child rearing by simply assuming that a component of child-rearing cost is proportional to the externality factor $G(L_t)$. Parents' time cost is another component of child-rearing. Assuming that each child requires a fixed amount, ρ, of parent's time, the third factor in (14) represents parent's foregone earnings as the time cost.

The following proposition characterizes the underlying dynamical system of our endogenous growth model.

Proposition 1. *Starting from any initial condition k_0, L_0 an economy $\mathscr{E} = \langle U, f, G \rangle$ satisfying assumptions A:1–A:2, (13) and (14) has a unique competitive equilibrium path given by the solution of the following system of bi-variate iterative maps:*

$$k_{t+1} = \Psi\left(\frac{\theta + [\gamma + \rho w(k_t)]G(L_t)}{a}\right) \tag{15}$$

$$\frac{L_{t+1}}{L_t} = \frac{(1-\delta)(1-a)w(k_t)G(L_t)}{[\theta + [\gamma + \rho w(k_t)]G(L_t)] + \Psi\left(\frac{\theta + [\gamma + \rho w(k_t)]G(L_t)}{a}\right)} \tag{16}$$

k_0 and L_0 are given.

Proof: Follows immediately from Lemma 2, (10), (13) and (14).

The existence of competitive equilibrium path could be shown following the argument in Raut [1991]. Our purpose is to examine the global dynamics of the above system of difference equations. The present state of knowledge about the global dynamics of bivariate iterative maps are limited. In the past few decades, however, much has been established about the complex global dynamic behaviors of one-dimensional iterative maps. In the next section, we use these results in studying the global dynamics of our model by reducing the fundamental difference equations (15)–(16) to a univariate difference equation by assuming that child rearing cost is fixed over time. In the following section we demonstrate with examples the complex dynamic properties of the above bivariate system for general child-rearing cost functions.

3. Constant child rearing cost, $\theta_t = \theta \; \forall t \geq 1$

Assumption A:3 (Constant child rearing cost). *$\gamma = \rho = 0$ in equation (14), i.e., constant child rearing cost, $\theta_t = \theta$, $\forall t \geq 1$.*

Under assumption A:3 we have from equation (15) that $k_{t+1} = k^*$, a constant, for all $t \geq 0$. (16) now reduces to

$$L_{t+1} = \lambda L_t G(L_t) \equiv h(L_t) \tag{17}$$

where $\lambda = \dfrac{(1-\delta)(1-a)w^*}{\theta + k^*}$ and $w^* = f(k^*) - k^* f'(k^*)$. Note that for this economy the difference equation (17) characterizes the competitive equilibrium path completely.

The dynamic properties of the competitive equilibrium path depends on the nature of external effect, $G(L_t)$, that population size has on productivity level. Suppose this relationship is negative for all population sizes and after rescaling of population size appropriately suppose this relationship is simply represented by a linear function of the from, $G(L_t) = \zeta(1 - L_t)$, $\zeta > 0$. For this specification (17) reduces to the well-known logistic map, $L_{t+1} = \mu L_t(1 - L_t)$, $\mu = \lambda\zeta > 0$. As is well known the logistic map generates complicated dynamics for various values of μ, see for instance (Devaney [1989]).

It is more reasonable to assume that at low levels of population density, $G(L_t)$ is an increasing function of L_t, and at high levels of population density, $G(L_t)$ is a decreasing function of L_t. We represent this relationship by the following functional form:

$$G(L) = \zeta e^{-(L-\bar{L})^2/2}. \tag{18}$$

Substituting equation (18) in equation (17) we have the following iterative map:

$$x_{t+1} = h_\mu(x_t), \quad \text{where} \quad h_\mu(x) \equiv \mu x e^{-(x-\bar{L})^2/2}, \quad \mu = \lambda\zeta. \tag{19}$$

We further assume that $1 < \mu < e^{\bar{L}^2/2}$. Not much is known about the dynamic properties of the above map, which we briefly investigate now.

3.1 Period doubling bifurcations of $h_\mu(\cdot)$

In this section we examine the qualitative properties of the periodic and fixed points of our dynamical system, (19), as we increase the value of μ from $\mu = 1$. Denote by $h_\mu^t(x) \equiv \overbrace{h_\mu \circ h_\mu \circ \cdots h_\mu(x)}^{t \text{ times}}$. We will sometimes denote by $x_t = h_\mu^t(x_0)$, $t \geq 0$ with $x_0 = h_\mu^0(x_0)$. The shape of the phase diagram of h_μ depends on the parameter values $\bar{L} > 0$ and $\mu > 1$. A typical phase diagram of the dynamic system in (19) is shown in Figure 1. It is clear that $x = 0$ is a locally stable steady-state and there are two other steady-states, $L^* = \bar{L} - \sqrt{2\log\mu}$ and $L^{**} = \bar{L} + \sqrt{2\log\mu}$.

Define the interval $S \equiv [L^*, \hat{L}]$, where $\hat{L}$ is such that $\hat{L} > L^*$ and $h_\mu(\hat{L}) = L^*$. Note that this interval varies with μ. $L^\# \in S$ is a *critical point* of h_μ if $h'_\mu(L^\#) = 0$. It is easily seen that the function $h_\mu(x)$ has only one critical point, $L^\#$, given by

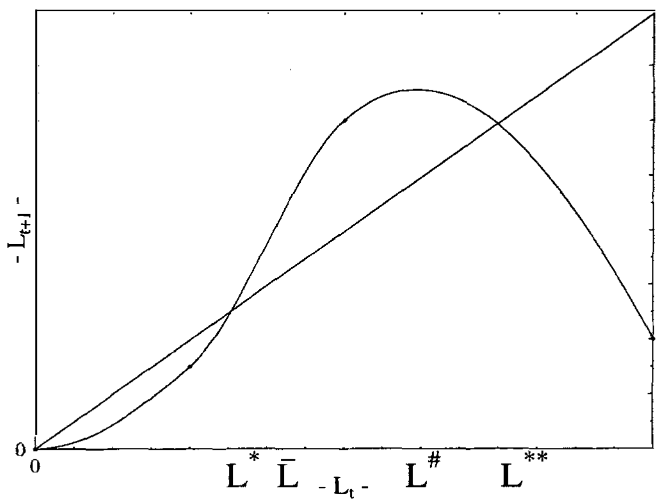

Figure 1. Phase diagram of $h_\mu(L_t)$.

$L^{\#} = 0.5(\bar{L} + \sqrt{\bar{L}^2 + 4})$. Position of the critical point relative to L^{**} is critical in determining the nature of the dynamic equilibrium path. For instance, if $L^{\#} > L^{**}$, a dynamic equilibrium path is monotonic around the steady-state L^{**}, and in the interval $(L^*, L^{**}]$; if $L^{\#} < L^{**}$, equilibrium path exhibits fluctuations around the steady-state L^{**}; if $L^{\#}$ coincides with L^{**}, then if the system starts to the left of L^{**}, it is monotonic, and if the system starts to the right of L^{**}, it snaps back to the left of L^{**} in the next period and remains monotonic thereafter.

The dynamics of $h_\mu(x)$ outside S is completely known: $h_\mu^n(x)$ converges to the trivial fixed point 0 as $n \to \infty$. Moreover, note that since $h_\mu \circ h_\mu(L^{\#}) < L^{\#}\ \forall \mu > 1$, we have that $h_\mu(S) \subseteq S$. Hence S and $\Re_+ - S$ are invariant sets under the iterations of the map $h_\mu(\cdot)$. Much of the interesting dynamic phenomena of the system (19) occurs in the invariant set S. Thus to study the dynamics of the map h_μ, without loss of generality, we restrict to the invariant state space S.

We begin with defining a few concepts from the literature of one dimensional discrete dynamical systems. The *forward orbit* or *dynamic path* of the dynamic system starting at x is the set $\{h_\mu^t(x)\}_0^\infty$. A *fixed point* of $h_\mu(x)$ is a $p \in S$ such that $h_\mu(p) = p$. A *periodic point of period n* of the map $h_\mu(x)$ is a fixed point $p \in S$ of the map $g_\mu(x) \equiv h_\mu^n(x)$, i.e., $h_\mu^n(p) = p$ for the smallest integer $n \geq 0$; the corresponding orbit $\{h_\mu^t(p)\}_0^n$ is called a *periodic orbit*. A fixed point $p \in S$ is *locally stable* (*attracting, an attractor, or sink*) if $|h'_\mu(p)| < 1$, *unstable* (*repelling, a repeller, or a source*) if $|h'_\mu(p)| > 1$, and *non-hyperbolic* if $|h'_\mu(p)| = 1$; $h'_\mu(p)$ is known as the *multiplier* of the fixed point p. If the economy starts in the neighborhood of an attracting periodic point, then in the long-run the economy converges to the periodic orbit associated with the periodic point. We are interested in knowing how many attracting periodic orbits are there for different values of μ, and how the stability properties of the periodic points change as we increase the value of μ. For this purpose, the notion of Schwartzian derivative is very useful:

Schwartzian derivative, $Sh_\mu(x)$ of $h_\mu(x)$ is defined as follows:

$$Sh_\mu(x) \equiv \frac{h'''_\mu(x)}{h'_\mu(x)} - \frac{3}{2}\cdot\left[\frac{h''_\mu(x)}{h'_\mu(x)}\right]^2$$
$$= -2\left[\frac{1}{2} - \frac{(x-\bar{L})^2}{2} + \frac{1+(x-\bar{L})(\bar{L}-2x)}{1-x(x-\bar{L})}\right] - \frac{3}{2}\cdot\left[\frac{\bar{L}-2x}{1+\bar{L}x-x^2} - x - \bar{L}\right]^2$$
$$\textit{for all} \quad x \neq L^{\#}.$$

The above is independent of μ. Note that it is not possible to determine the sign of $Sh_\mu(x)$ for all x, since it also depends on $\bar{L}$. We fix $\bar{L} = 2$ in our analysis, and compute $Sh_\mu(x)$ numerically and find that $Sh_\mu(x) < 0\ \forall x \in S,\ x \neq L^{\#}$.

The *stable set of* a periodic point $p \in S$ is defined as $W^s(p) \equiv \{x \in S \mid \lim_{t\to\infty} (h_\mu^m)^t(x) = p\}$.

Since h_μ has one critical point, and since Schwartzian derivative $Sh_\mu(x) < 0$ $\forall x \in S, x \neq L^{\#}$, from Devaney [1989, Lemma 11.7, p. 72] it follows that h_μ has finitely many periodic points of period m for any integer $m \geq 1$; and from Devaney [1989, Theorem 11.4, p. 71] it follows that h_μ has at most 3 attracting periodic orbits. Since for any periodic point $p \in S$, $W^s(p)$ is a subset of the bounded set S, $W^s(p)$ is also bounded. Hence it follows from the remarks in Devaney [1989, pp. 73–74] that h_μ

has, in fact, at most one attracting periodic orbit. We summarize these properties of h_μ in the following proposition:

Proposition 2. *For $\mu > 1$, $h_\mu(x)$ has finitely many periodic orbits of period m for any integer $m \geq 1$ and at most one of these periodic orbits is attracting.*

For given $\bar{L} > 0$, high values of μ will move L^{**} to the right of $\bar{L}$. Note that

$$h'_\mu(L^{**}) = 1 - \sqrt{2\log\mu}(\bar{L} + \sqrt{2\log\mu})$$

$$h'_\mu(L^{*}) = 1 + \sqrt{2\log\mu}(\bar{L} - \sqrt{2\log\mu})$$

Since $\bar{L} - \sqrt{2\log\mu} > 0$, it is clear that the multiplier of the fixed point L^* is always >1, and hence the fixed point L^* is always repelling. Let us examine the stability property of L^{**} as we increase μ. Note that $h'_\mu(L^{**})$ is a decreasing function of $\mu > 1$; for μ close to one, $h'_\mu(L^{**}) < 1$ and for a higher value of μ, $h'_\mu(L^{**}) = 0$ and for even higher values of μ, $h'_\mu(L^{**})$ becomes ≤ -1. The parameter value $\mu = \mu_1$ at which $h'_{\mu_1}(L^{**}) = -1$, i.e., at which L^{**} becomes non-hyperbolic, plays an important role: for $\mu \geq \mu_1$, L^{**} ceases to be an attracting fixed point. It will be convenient to denote this non-hyperbolic fixed point as L_1^{**}. It can be easily seen that the parameter value $\mu_1 > 1$ is obtained as the solution of the equation $\bar{L} = \dfrac{2(1-\log\mu)}{\sqrt{2\log\mu}}$. In the rest of the paper whenever we refer to any numerical calculations, we fix the value of $\bar{L} = 2$. For this $\bar{L}$, one can easily compute the following:

$$\mu_1 = 1.307281; \quad L_1^{**} = 2.732051; \quad \left.\frac{\partial (h_\mu^2)'(L_1^{**})}{\partial\mu}\right|_{\mu=\mu_1} = 7.239 > 0.$$

Thus by applying Devaney [1989, Theorem 12.7][1], there exists an interval I containing L_1^{**} and a function $m(L)$ relating each $L \in I$ to a μ with the property that $h_{m(L)}(L) \neq L$ but $h^2_{m(L)}(L) = L$. Or in otherwords, suppose $\check{L} \in I$, $\check{L} \neq L_1^{**}$, and let $\check{\mu} = m(\check{L})$, then at $\mu = \check{\mu}$, $\check{L}$ is not a fixed point of $h_{\check{\mu}}$ but it is a periodic point of period 2 of the map $h_{\check{\mu}}$. Moreover, as shown in Devaney [1989, p. 91],

$$m''(L_1^{**}) = \frac{-\frac{2}{3} Sh_{\mu_1}(L_1^{**})}{\left.\frac{\partial}{\partial\mu}\right|_{\mu=\mu_1} (h_\mu^2)'(L_1^{**})}.$$

Since $Sh_\mu(L) < 0 \; \forall L \in S$ and the denominator is strictly positive, we have that $m(L)$ is convex to the L-axis as shown in Figure 2.

The results are stated more precisely in the following proposition:

Proposition 3. *Let $L_1^{**} \equiv L^{**}(\mu_1)$. For $1 < \mu < \mu_1$, $L^{**}(\mu)$ is the only attracting periodic point of h_μ. Moreover, there exists an interval I containing L_1^{**} and a function $m: I \to \Re$, convex to the L-axis, such that $m(L) \geq \mu_1$ for all $L \in I$ and for all $L \in I$, $L \neq L_1^{**}$,*

$$h_{m(L)}(L) \neq L \quad \text{but } h^2_{m(L)}(L) = L.$$

[1] To apply Devaney's theorem, the following change of coordinates can be used $z = x - L_1^{**}$ and $g_\mu(z) = h_\mu(z + L_1^{**}) - L_1^{**}$.

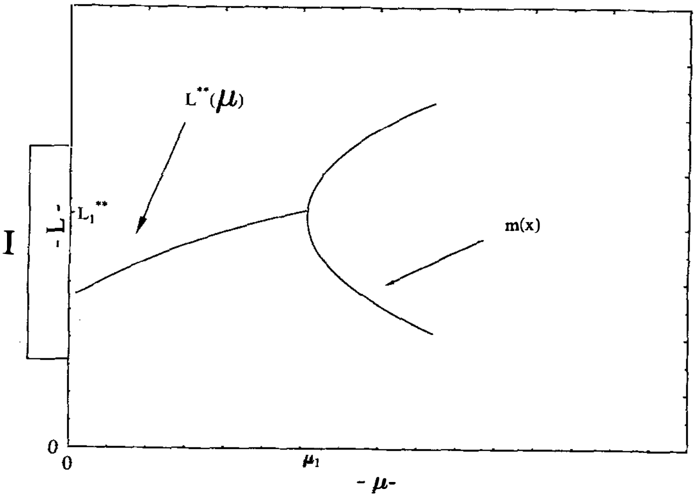

Figure 2. Period doubling bifurcation of $h_\mu(L_t)$.

It is hard to derive the multipliers of the periodic points of period 2 for h_μ to see if they are stable; when they are stable, it is even harder to calculate the parameter value μ_2 at which these periodic points become non-hyperbolic and go through period doubling bifurcations. However, for each discrete value of μ at step of 0.0001 we have numerically computed the dynamic paths for 1000 periods starting at different initial values x_0; the last 300 values of each path are plotted against μ in Figure 3. This shows complicated bifurcations of the map $h_\mu(x)$.

Thus from our similation exercise we find that h_μ follows the similar period doubling bifurcation phenomenon as the widely studied logistic map. To sum up,

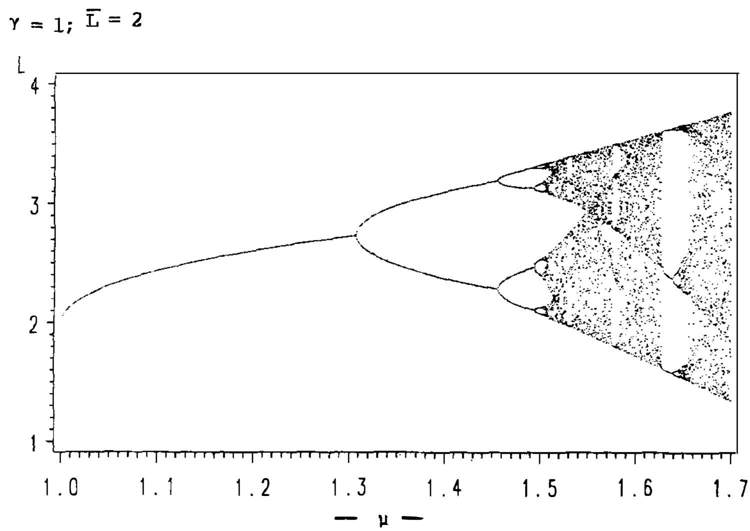

Figure 3. Bifurcation of population dynamics.

we find that for $1 < \mu < 1.307281 = \mu_1$, the fixed point $L^{**}(\mu)$ is the only attracting periodic point. At $\mu = 1.307281$, the fixed point $L^{**}(\mu)$, ceases to be attracting, and two attracting periodic points of period two are born, they are attracting until they bifurcate at $\mu = 1.4558$ and four stable periodic points of period four are born, and so on.

For most unimodal maps the above type of period doubling bifurcations is a route to chaos. In the next section we examine if h_μ also exhibits chaotic behavior.

3.2 Chaotic dynamics of $h_\mu(\cdot)$

Li and Yorke [1975] have shown that if a map h_μ has a periodic point of period three, or more generally, if h_μ admits a point $\check{L}\in S$ such that the Li–Yorke condition,

$$h_\mu^3(\check{L}) \leq \check{L} < h_\mu(\check{L}) < h_\mu^2(\check{L})$$

is satisfied, then h_μ exhibits chaotic behavior in the sense that h_μ has periodic orbits of any integer period, there exists a uncountable set, W, of non-periodic states such that a pair of orbits orginating from any two initial states in W move apart and come close to each other infinitely often, and an orbit originating in W does not converge to any periodic orbit. (See Baumol and Benhabib [1989], Majumdar and Mitra [1992] and Nishimura and Yano [1994] for a statement and for other applications of the result.)

In Figure 4 we have drawn the graphs of $h_\mu^3(x)$, $h_\mu^2(x)$ and $h_\mu(x)$ for $\mu = 1.65$. Figure 4 shows that the graph of h_μ^3 intersects the 45° line, and hence has a periodic point of period three.

For another example, take $\mu = 1.70$, and $\check{L} = 1.56$, then it could be easily shown that $h_\mu^3(\check{L}) = 1.34$, $h_\mu(\check{L}) = 2.41$, $h_\mu^2(\check{L}) = 3.77$ which satisfy Li–Yorke condition. Thus the map h_μ exhibits chaotic behavior for various values of μ.

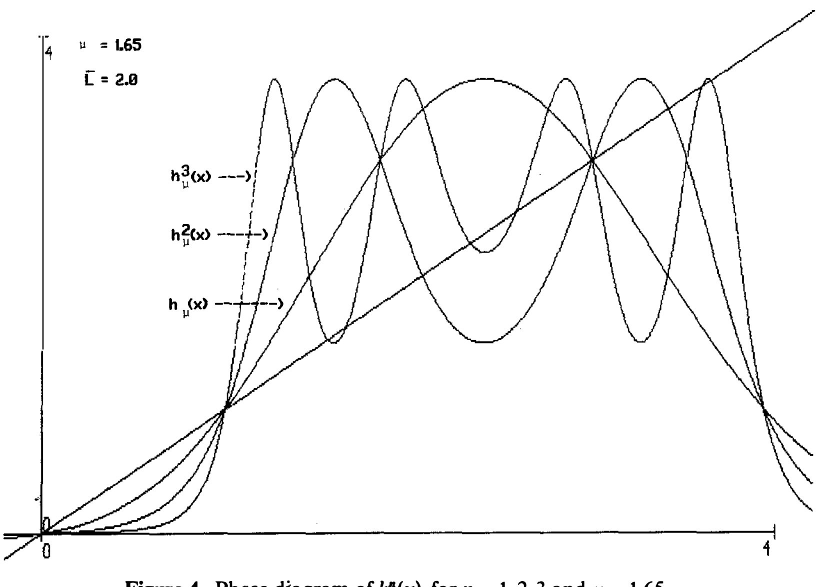

Figure 4. Phase diagram of $h_\mu^n(x)$, for $n = 1, 2, 3$ and $\mu = 1.65$.

Moreover, from the period doubling bifurcation diagram of h_μ in Figure 3, it is clear that starting with a stable periodic orbit of period three for a parameter value somewhere in the neighborhood of $\mu = 1.63$, the system goes through a sequence of period doubling bifurcations as we increase the parameter value μ.

It is easy to see from (17) that appropriate specifications of $G(L_t)$ can generate equilibrium paths with steady exponential growth in population.

4. Time-varying child rearing cost

In the literature on endogenous fertility, it is usual to assume that the cost of child rearing equals the wages foregone by the parent on the time spent on rearing. Unless the time varies inversely with the wage rate, the cost of child rearing will vary over time. In our model, even if we assume the cost to be constant, we find that the equilibrium wage rate is not constant. To be consistent, child rearing cost should be allowed to vary. Besides doing so also yields interesting dynamics. To simplify our analysis we assume that the production function is from the following Cobb–Douglas class:

$$f(k) = k^\sigma, \quad 0 < \sigma < 1. \tag{20}$$

Assuming further that $\theta = 0$, the bivariate system of difference equations, (15)–(16) reduces to the system:

$$k_{t+1} = \frac{\sigma G(L_t)[\gamma + \rho(1-\sigma)k_t^\sigma]}{a(1-\sigma)} \tag{21}$$

$$\frac{L_{t+1}}{L_t} \equiv n_t = \left[\frac{(1-a)(1-\delta)}{\rho}\right]\left[\frac{a(1-\sigma)}{a(1-\sigma)+\sigma}\right]\left[\frac{\rho(1-\sigma)k_t^\sigma}{\gamma+\rho(1-\sigma)k_t^\sigma}\right]. \tag{22}$$

Note that $n_t < \frac{(1-a)(1-\delta)}{\rho}$. Thus whenever $\frac{(1-a)(1-\delta)}{\rho} < 1$, it is seen from equation (22) that the working population converges to zero as $t \to \infty$, regardless of the process $G(L_t)$.

Notice that if child rearing involves only the time cost, i.e., $\gamma = 0$, then equilibrium fertility rate, n_t is constant, and the dynamics of the system is determined by the one dimensional iterative map in (21) which, however, is not constant over time. We first analyze the dynamics of the equilibrium assuming $\gamma = \theta = 0$.

From (22) it follows that n_t, the growth in working population, is a constant $n^* = \frac{(1-a)(1-\delta)a(1-\sigma)}{\rho[a(1-\sigma)+\sigma]}$ (independent of the process $G(L_t)$) so that $L_t = L_0(n^*)^t$

From (21) we note that

$$\log k_{t+1} = \log G(L_t) + \log\sigma + \log\rho - \log a + \log k_t. \tag{23}$$

Denoting $\log k_t$ by x_t, $\log G(L_t)$ by g_t and $\log\sigma + \log\rho - \log a$ by ω, the solution to (23) is

$$x_{t+1} = \frac{\omega(1-\sigma^{t+1})}{(1-\sigma)} + x_0\sigma^t + \sum_{\tau=0}^{t} \sigma^\tau g_{t-\tau}. \tag{24}$$

4.1. No long-run growth

Suppose $G(L) = [\alpha + \beta e^{-\zeta L}]^{-1}$, $\alpha > 0$, $\beta > 0$, and $\zeta > 0$ then $G(L)$ is bounded and converges to $1/\alpha$ or $1/(\alpha + \beta)$ depending on whether $L_t \to \infty$ (i.e., $n^* > 1$) or $L_t \to 0$ (i.e., $n^* < 1$) as $t \to \infty$. Hence using (24) and noting that $0 < \sigma < 1$, we can say that as $t \to \infty$, x_t converges in either case. The average and marginal product of labor, and hence the welfare of each member of a generation, also converge to constants, with the working population increasing indefinitely in the first case and dwindling to zero in the second. More generally, if $G(L) > 0$ is bounded above, then (24) implies that $(1 - \sigma)x_t$ is bounded above as well, so that the welfare of each member of any generation is bounded above, with the size of the working population growing or dwindling depending on whether n^* is greater or less than 1.

4.2. Super-exponential growth

Assume the parameters to be such that $n^* > 1$ (so that $L_t \to \infty$ as $t \to \infty$) and $G(L_t)$ is unbounded. Suppose $G(L_t)$ behaves (for large values of L_t) as $e^{\mu L_t}$, $\mu > 0$ so that g_t behaves as $\mu L_t = \mu L_0(n^*)^t$. Then from (24) it follows that x_{t+1} (for large values of t) is

$$\frac{\omega(1 - \sigma^{t+1})}{1 - \sigma} + x_0\sigma^t + \mu L_0(n^*)^t \frac{\left(1 - \frac{\sigma}{n^*}\right)^{t+1}}{1 - \frac{\sigma}{n^*}}. \tag{25}$$

Since $n^* > 1 > \sigma$, as $t \to \infty$, x_{t+1} behaves as $\dfrac{\mu L_0(n^*)^t}{1 - \sigma/n^*} + \dfrac{\omega}{1 - \sigma}$. This in turn means that k_t behaves asymptotically as $\exp\left[\dfrac{\mu L_0(n^*)^t}{1 - \sigma/n^*}\right]$ and the average product of labor $= G(L_t)k_t^\sigma$ behaves as $e^{(\sigma + \nu L_0)(n^*)^t}$, where ν is a positive constant! Thus one obtains super-exponential growth.

4.3. Exponential growth

Most endogenous growth models are geared to generate long-run exponential growth in per capita output. To generate long-run exponential growth in our model, suppose $G(L_t)$ behaves like $A(L_t)^\mu$ for large values of L_t ($\mu > 0$ implies that $G(L_t)$ is still unbounded); then g_t behaves like $\mu \log L_t = \mu[\log L_0 + tn^*]$. From (24) it can be shown that x_{t+1} behaves as $\dfrac{\mu n^* t}{1 - \sigma}$ for large values of t. This means that k_t grows exponentially at the rate $\dfrac{\mu n^*}{1 - \sigma}$. With $G(L_t)$ behaving like $[L_0(n^*)^t]^\mu = (L_0)^\mu(n^*)^{\mu t}$, exponential growth of k_t implies exponential growth in the average and marginal product of labor and in the welfare of each member of a generation.

4.4 Child rearing cost proportional to the externality effect and super-exponential growth

Assume that the child-rearing cost is proportional to the externality factor $G(L_t)$ and other types of costs are absent by assuming that $\rho = \theta = 0$. For this type of child rearing cost, we show that the specification of $G(L) = AL^{\mu}$ that led to stable exponential growth in the previous subsection now leads to super-exponential growth for almost all initial k_0 and L_0.

Note that (21) in this case becomes

$$k_{t+1} = \frac{\sigma\gamma}{(1-\sigma)a} G(L_t) \tag{26}$$

using the above in equation (22), we have

$$L_{t+1} = \frac{(1-\delta)a(1-a)(1-\sigma)^2}{[a(1-\sigma)+\sigma]\gamma} \left[\frac{\sigma\gamma G(L_{t-1})}{(1-\sigma)a}\right]^{\sigma} L_t. \tag{27}$$

Substituting $G(L_t) = AL_t^{\mu}$ in (27), and taking logarithms of both sides, defining $l_{t+1} \equiv \log L_{t+1}$ and $\bar{w} \equiv \log\left(\frac{(1-\delta)a(1-a)(1-\sigma)^2}{[a(1-\sigma)+\sigma]\gamma} \cdot \frac{\sigma A\gamma}{(1-\sigma)a}\right)$, we get:

$$l_{t+1} - l_t - \mu\sigma l_{t-1} = \bar{w}. \tag{28}$$

The solution of (28) is

$$l_t = -\frac{\bar{w}}{\mu\sigma} + D_1 v_1^t + D_2 v_2^t. \tag{29}$$

The initial conditions $D_1 + D_2 - \frac{\bar{w}}{\mu\sigma} = \log L_0$ and $D_1 v_1 + D_2 v_2 - \frac{\bar{w}}{\mu\sigma} = \log L_0 + \log n_0$ determine D_1 and D_2. Of course n_0 depends on the given k_0. Solving for D_1 and D_2, we get:

$$D_1 = \left[\left(\frac{\bar{w}}{\mu\sigma} + \log L_0\right)(1 - v_2) + \log n_0\right] \frac{1}{\sqrt{1+4\mu\sigma}} \tag{30}$$

$$D_2 = \left[\left(\frac{\bar{w}}{\mu\sigma} + \log L_0\right)(v_1 - 1) - \log n_0\right] \frac{1}{\sqrt{1+4\mu\sigma}}. \tag{31}$$

For small values of $\mu\sigma$, $\sqrt{1+\mu\sigma} \approx 1 + 2\mu\sigma$, so that we see from (29) that $l_t = \log L_t$ grows asymptotically at the rate $\mu\sigma$. From equation (27), it follows that $\log k_{t+1}$ also grows at the same rate as well. Hence k_t and L_t grow super-exponentially.

Consider the initial values of L_0 and k_0 given by:

$$\frac{(1-\delta)a(1-a)(1-\sigma)^2}{[a(1-\sigma)+\sigma]\gamma} k_0^{\sigma} = 1 \tag{32}$$

$$\frac{\sigma\gamma}{(1-\sigma)a} G(L_0) = k_0. \tag{33}$$

It is clear then from repeated application of (26) and (27) that k_t and L_t remain at L_0 and k_0 so that these are steady-state values. Further, any values other than these will lead to either $L_1 \neq L_0$ or $k_1 \neq k_0$ or both so that the economy will not be in a steady state. Since $G(L)$ is monotonic, (32) and (33) produce unique steady state of the model. Noting that $G(L) = AL^{\mu}$, equations (32) and (33) imply $\bar{w} = -\mu\sigma \log L_0$ and $n_0 = 1$ so that from equations (30) and (31) we find $D_1 = D_2 = 0$ so that the economy remains in a steady state from period zero.

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